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Determinantal martingales and noncolliding diffusion processes

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Abstract

Two aspects of noncolliding diffusion processes have been extensively studied. One of them is the fact that they are realized as harmonic Doob transforms of absorbing particle systems in the Weyl chambers. Another aspect is integrability in the sense that any spatio-temporal correlation function can be expressed by a determinant. The purpose of the present paper is to clarify the connection between these two aspects. We introduce a notion of determinantal martingale and prove that, if the system has determinantal-martingale representation, then it is determinantal. In order to demonstrate the direct connection between the two aspects, we study three processes.

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1. Introduction

Two aspects of noncolliding diffusion processes have been extensively studied in probability theory and random matrix theory. One of them is the fact that, although they are originally introduced as eigenvalue processes of matrix-valued diffusions [9,5], they are realized as

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harmonic Doob transforms of absorbing particle systems in the Weyl chambers [15,29,21]. It implies that the noncolliding particle systems can be regarded as multivariate extensions of the three-dimensional Bessel process, which is realized as an h-transform of absorbing Brownian motion in one-dimension with an absorbing wall at the origin [25]. Another aspect is integrability in the sense that any spatio-temporal correlation function can be explicitly expressed by a determinant specified by a single continuous function called the correlation kernel. Such determinantal processes are considered to be dynamical extensions of random matrix models in which the eigenvalue ensembles provide determinantal point processes [28,12,50]. The purpose of the present paper is to clarify the connection between these two aspects. We introduce a notion of *determinantal martingale* and prove that, if the interacting particle system has an expression called *determinantal-martingale representation (DMR)*, then it is determinantal. In order to demonstrate the direct connection between the two aspects through DMR, we study three processes, the noncolliding Brownian motion (Dyson's Brownian motion model with $\beta = 2$), the noncolliding squared Bessel process, and the noncolliding Brownian motion on a circle.

We consider a continuous time Markov process $Y(t),\ t \in [0, \infty)$ on the state space S which is a connected open set in $\mathbb{R}$. It is a diffusion process in S or a process showing a position on the circumference $S = [0, 2\pi r)$ of a diffusion process moving around on the circle with a radius $r > 0$; $\mathrm{S}^1(r) \equiv \{x \in [0, 2\pi r) : x + 2\pi r = x\}$. The probability space is denoted by $(\Omega, \mathcal{F}, \mathrm{P}_v)$ with expectation E_v, when the initial state is fixed to be $v \in S$. When v is the origin, the subscript is omitted. We introduce a filtration $\{\mathcal{F}(t) : t \in [0, \infty)\}$ generated by Y so that it satisfies the usual conditions (see, for instance, p. 45 in [42]). We assume that the process has a transition density (TD), $p(t, y|x),\ t \in [0, \infty),\ x, y \in S$ such that for any measurable bounded function $f(t, x),\ t \in [0, \infty),\ x \in S$,

$$\mathrm{E}[f(t, Y(t))|\mathcal{F}(s)] = \int_S dy\, f(t, y) p(t - s, y|Y(s)) \quad \text{a.s.}, 0 \le s \le t < \infty. \tag{1.1}$$

For $N \in \mathbb{N} \equiv \{1, 2, \dots\}$, we put

$$\mathbb{W}_N(S) = \{\boldsymbol{x} = (x_1, \dots, x_N) \in S^N : x_1 < x_2 < \dots < x_N\}.$$

If $S = \mathbb{R}$, $\mathbb{W}_N(\mathbb{R})$ is the *Weyl chamber* of type A_{N-1} [15]. For $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N(S)$, we define a measure ξ by a sum of point masses concentrated on $u_j,\ 1 \le j \le N$; $\xi(\cdot) = \sum_{j=1}^N \delta_{u_j}(\cdot)$. Depending on ξ, we assume that there is a one-parameter family of continuous functions

$$\mathcal{M}_\xi^v(\cdot, \cdot) : [0, \infty) \times S \mapsto \mathbb{R}$$

with parameter $v \in S$, such that the following conditions are satisfied.

(M1) $\mathcal{M}_\xi^{u_k}(t, Y(t)),\ 1 \le k \le N, t \in [0, \infty)$ are continuous-time martingales;

$$\mathrm{E}[\mathcal{M}_\xi^{u_k}(t, Y(t))|\mathcal{F}(s)] = \mathcal{M}_\xi^{u_k}(s, Y(s)) \quad \text{a.s. for all } 0 \le s \le t.$$

(M2) For any time $t \ge 0$, $\mathcal{M}_\xi^{u_k}(t, x),\ 1 \le k \le N$ are linearly independent functions of x.

(M3) For $1 \le j, k \le N$,

$$\lim_{t \downarrow 0} \mathrm{E}_{u_j}[\mathcal{M}_\xi^{u_k}(t, Y(t))] = \delta_{jk}.$$

We call $\mathcal{M}_\xi(t, x) = \{\mathcal{M}_\xi^v(t, x)\}$ martingale functions.