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A strictly stationary β-mixing process satisfying the central limit theorem but not the weak invariance principle

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Abstract

In 1983, N. Herrndorf proved that for a ϕ-mixing sequence satisfying the central limit theorem and $\liminf_{n\to\infty} \sigma_n^2/n > 0$, the weak invariance principle takes place. The question whether for strictly stationary sequences with finite second moments and a weaker type (α, β, ρ) of mixing the central limit theorem implies the weak invariance principle remained open.

We construct a strictly stationary β-mixing sequence with finite moments of any order and linear variance for which the central limit theorem takes place but not the weak invariance principle.

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Keywords: Central limit theorem; Invariance principle; Mixing conditions; Strictly stationary process

1. Introduction and notations

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If $T\colon \Omega \to \Omega$ is one-to-one, bi-measurable and measure preserving (in the sense that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$), then the sequence $\left(f \circ T^k\right)_{k\in\mathbb{Z}}$

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is strictly stationary for any measurable $f: \Omega \to \mathbb{R}$. Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable $f: \Omega \to \mathbb{R}$, we define $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$ and $S_n^*(f,t) := S_{\lfloor nt \rfloor}(f) + (nt - \lfloor nt \rfloor) f \circ T^{\lfloor nt \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer which is less than or equal to x.

We say that $(f \circ T^j)_{j \geqslant 1}$ satisfies the *central limit theorem with normalization* a_n if the sequence $(a_n^{-1} S_n(f))_{n \geqslant 1}$ converges weakly to a standard normal distribution. Let $C[0,1]$ denote the space of continuous functions on the unit interval endowed with the norm $\|g\|_\infty := \sup_{t \in [0,1]} |g(t)|$.

Let $D[0,1]$ be the space of real valued functions which have left limits and are continuous-from-the-right at each point of $[0,1)$. We endow it with Skorohod metric (cf. [2]). We define $S_n^{**}(f,t) := S_{\lfloor nt \rfloor}(f)$, which gives a random element of $D[0,1]$.

We shall say that the strictly stationary sequence $\left(f \circ T^j\right)_{j \geqslant 0}$ *satisfies the weak invariance principle in* $C[0,1]$ *with normalization* a_n (respectively *in* $D[0,1]$) if the sequence of $C[0,1]$ (of $D[0,1]$) valued random variables $\left(a_n^{-1} S_n^*(f,\cdot)\right)_{n \geqslant 1}$ (resp. $\left(a_n^{-1} S_n^{**}(f,\cdot)\right)_{n \geqslant 1}$) weakly converges to a Brownian motion process in the corresponding space.

Let $\mathcal{A}$ and $\mathcal{B}$ be two sub-σ-algebras of $\mathcal{F}$, where $(\Omega, \mathcal{F}, \mu)$ is a probability space. We define the α-mixing coefficients as introduced by Rosenblatt in [14]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \left\{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \right\}.$$

Define the β-mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} \left| \mu(A_i \cap B_j) - \mu(A_i)\mu(B_j) \right|,$$

where the supremum is taken over the finite partitions $\{A_i, 1 \leqslant i \leqslant I\}$ and $\left\{B_j, 1 \leqslant j \leqslant J\right\}$ of Ω of elements of $\mathcal{A}$ (respectively of $\mathcal{B}$). They were introduced by Volkonskii and Rozanov [16].

The ρ-mixing coefficients were introduced by Hirschfeld [8] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \left\{ |\mathrm{Corr}(f,g)|, \ f \in \mathbb{L}^2(\mathcal{A}), \ g \in \mathbb{L}^2(\mathcal{B}) \right\},$$

where $\mathrm{Corr}(f,g) := [\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)] \left[\|f - \mathbb{E}(f)\|_{\mathbb{L}^2} \|g - \mathbb{E}(g)\|_{\mathbb{L}^2} \right]^{-1}$.

Ibragimov [9] introduced for the first time ϕ-mixing coefficients, which are given by the formula

$$\phi(\mathcal{A}, \mathcal{B}) := \sup \left\{ |\mu(B \mid A) - \mu(B)|, \ A \in \mathcal{A}, \ B \in \mathcal{B}, \ \mu(A) > 0 \right\}.$$

The coefficients are related by the inequalities

$$2\alpha(\mathcal{A}, \mathcal{B}) \leqslant \beta(\mathcal{A}, \mathcal{B}) \leqslant \phi(\mathcal{A}, \mathcal{B}), \qquad \alpha(\mathcal{A}, \mathcal{B}) \leqslant \rho(\mathcal{A}, \mathcal{B}) \leqslant 2\sqrt{\phi(\mathcal{A}, \mathcal{B})}. \tag{1}$$

For a strictly stationary sequence $(X_k)_{k \in \mathbb{Z}}$ and $n \geqslant 0$ we define $\alpha_X(n) = \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$ where $\mathcal{F}_u^v$ is the σ-algebra generated by X_k with $u \leqslant k \leqslant v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict). In the same way we define coefficients $\beta_X(n)$, $\rho_X(n)$, $\phi_X(n)$.

We say that the sequence $(X_k)_{k \in \mathbb{Z}}$ is α-*mixing* if $\lim_{n \to +\infty} \alpha_X(n) = 0$, and similarly we define β, ρ and ϕ-mixing sequences.

α, β and ϕ-mixing sequences were considered in the mentioned references, while ρ-mixing sequences first appeared in [12].