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Splitting multidimensional BSDEs and finding local equilibria

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Abstract

We introduce a new notion of local solution of backward stochastic differential equations (BSDEs) and prove that multidimensional quadratic BSDEs are locally but not globally solvable. Applied in a financial context on optimal investment, our results show that there exist local but no global equilibria when agents take both the absolute and the relative performance compared to their peers into account.

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1. Introduction

Backward stochastic differential equations (BSDEs) have recently become a central topic in probability theory and stochastic analysis, largely because of their importance in mathematical finance and other applications. A BSDE is of the form

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s,$$

where given are a d-dimensional Brownian motion W, an n-dimensional random variable ξ and a generator function f. A solution (Y, Z) consists of an n-dimensional semimartingale Y and

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an $(n \times d)$-dimensional control process Z predictable with respect to the filtration generated by W. Existence and uniqueness results have first been shown for BSDEs with generators f satisfying a Lipschitz condition; see for example Pardoux and Peng [10]. However, BSDEs like those in applications from mathematical finance typically involve generators f which are quadratic in the control variable Z. For such cases, Kobylanski [9] proved existence, uniqueness and comparison results when the terminal condition ξ is bounded and Y is one-dimensional $(n = 1)$. Subsequently, her results were generalized in different directions, such as to BSDEs with unbounded terminal conditions by Briand and Hu [2] and Delbaen et al. [3]. While Kobylanski's proof cannot be generalized to $n > 1$, Tevzadze [13] presents an alternative derivation of Kobylanski's results via a fix point argument. This yields as a byproduct an existence and uniqueness result also for $n > 1$ if the L^∞-norm of the terminal condition is sufficiently small. Yet another alternative proof for Kobylanski's results based on Malliavin calculus and allowing for BSDEs with delayed generators has recently been provided by Briand and Elie [1].

For a multidimensional quadratic BSDE (i.e., $n > 1$ and f is quadratic in the control variable Z), no general existence and uniqueness results are known. Frei and dos Reis [6] recently provided a counterexample to the existence of a solution, even with a bounded terminal condition ξ. Given this lack of global solvability, we introduce in this paper a new notion of local solution, which we call *split solution*. Its idea is to split the time interval $[0, T]$ into a finite number of subintervals and solve a suitable local form of the BSDE on every subinterval. We prove in Section 2 that there exists such a split solution if the terminal condition is in an appropriate space (BMO-closure of H^∞). We also provide counterexamples which show that even in this space, there does not exist a global solution, and even with a bounded terminal condition, there does not exist a split solution.

The new notion of split solution to multidimensional BSDEs nicely fits to the financial application, which we present in Section 3. As in Espinosa and Touzi [5] as well as Frei and dos Reis [6], we consider a model of a financial market where investors take not only their own absolute performance, but also the relative performance compared to their peers into account. We are interested in an equilibrium where every investor can find an individually optimal strategy. While Espinosa and Touzi [5] show the existence of such an equilibrium if all coefficients are deterministic, Frei and dos Reis [6] gave a counterexample to the existence in a stochastic situation. Our new result on split solutions allows us to establish in general the existence of local equilibria, namely equilibria over shorter time periods, while there does not exist an equilibrium over the whole time interval.

2. Splitting multidimensional BSDEs

After some preparation, we introduce in Section 2.2 the definition of split solution and show their existence. We then study in Section 2.3 how a multidimensional BSDE can be split in a minimal way.

2.1. Preparations

We work on a canonical Wiener space $(\Omega, \mathcal{F}_T, \mathbb{P})$ carrying a d-dimensional Brownian motion $W = (W^1, \dots, W^d)^\top$ restricted to the time interval $[0, T]$, and we denote by $(\mathcal{F}_t)_{t\in[0,T]}$ its augmented natural filtration. We use the following notation:

- $|z|^2 = \text{trace}(zz^\top)$ for $z \in \mathbb{R}^{n\times d}$,
- $\|M\|^2_{BMO} = \sup_\tau \left\| \mathbb{E}[\text{trace}\langle M\rangle_T - \text{trace}\langle M\rangle_\tau | \mathcal{F}_\tau] \right\|_{L^\infty}$ for an n-dimensional martingale M, where the supremum is over all stopping times τ valued in $[0, T]$,