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Large deviations of infinite intersections of events in Gaussian processes

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Abstract

Consider events of the form $\{Z_s \geq \zeta(s), s \in S\}$, where Z is a continuous Gaussian process with stationary increments, ζ is a function that belongs to the reproducing kernel Hilbert space R of process Z, and $S \subset \mathbb{R}$ is compact. The main problem considered in this paper is identifying the function $\beta^* \in R$ satisfying $\beta^*(s) \geq \zeta(s)$ on S and having minimal R-norm. The smoothness (mean square differentiability) of Z turns out to have a crucial impact on the structure of the solution. As examples, we obtain the explicit solutions when $\zeta(s) = s$ for $s \in [0, 1]$ and Z is either a fractional Brownian motion or an integrated Ornstein–Uhlenbeck process.

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1. Introduction

The large deviation principle (LDP) for Gaussian measures in Banach space, usually known as the (generalized) Schilder theorem, was established more than two decades ago by [3], see also

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[2,4]. In this LDP, a central role is played by the norm $\|f\|$ of paths f in the reproducing kernel Hilbert space of the underlying Gaussian process. More precisely, the probability of the Gaussian process being in some closed convex set A has exponential decay rate $-\frac{1}{2}\|f^*\|^2$, where f^* is the path in A with minimum norm, i.e., $\operatorname{argmin}_{f\in A}\|f\|$. The path f^* is called the "dominating point" in large deviations literature, and it has the interpretation of the *most probable path* (MPP) in A: if the Gaussian process happens to fall in A, with overwhelming probability it will be close to f^*.

Our interest in this topic stems from large deviation analyses of Gaussian queues, where various MPPs have been found explicitly. Addie et al. [1] consider a queueing system fed by a Gaussian process with stationary increments, and succeed in finding the MPP leading to overflow. This problem is relatively easy as the overflow event can be written as an infinite *union* of events $A = \cup_{t>0} A_t$, and the decomposition $\inf_{f\in A}\|f\| = \inf_{t>0}\inf_{f\in A_t}\|f\|$ applies. Here A_t corresponds to the event of overflow at time t, and due to the fact that finding the infimum over A_t turns out to be just a one-dimensional problem, the problem can be solved. In this paper we look at the intrinsically more involved situation where A is an *intersection*, rather than a union, of events: $A = \cap_t A_t$; decay rates, and the corresponding MPPs, of these intersections are then usually considerably harder to determine. In our setting the norm has to be minimized over a truly infinite-dimensional convex set in a Hilbert space.

Few results are known on MPPs of these infinite intersections of events. Norros [11] showed that the event of a queue with fractional Brownian motion (fBm) input having a busy period longer than, say, 1, corresponds to an infinite intersection of events; the set A consists of all f such that $f(t) \geq t$ for all $t \in [0, 1]$. However, the shape of the MPP in A remained an open problem. Interestingly, it was proven that the straight line, i.e., the path $f(t) = t$, is *not* optimal except for the Brownian case. In the case of Markovian input, the straight line is the asymptotically typical form for long busy periods [15, Thm. 11.24]. Mandjes and van Uitert [9, 10] analyzed buffer overflow in tandem, priority, and generalized processor sharing queues: it was shown that in these queues overflow relates to an infinite intersection of events, and explicit lower bounds on the minimizing norm (corresponding to upper bounds on the overflow probability) were obtained. Conditions were given under which this lower bound is tight — in that case obviously the path corresponding to the lower bound is also the MPP.

An important element in our analysis is the mean square smoothness of the Gaussian process involved. Most of our results assume that the process has the property that its local behaviors around any two distinct points are asymptotically independent, and this is shown to exclude smoothness. On the other hand, we show that the solution can have a simple finite-dimensional structure when the process is smooth.

This paper is organized as follows. Section 2 presents preliminaries on Gaussian processes and a number of other prerequisites. In Section 3 we focus on the most probable path in the set of paths f such that $f(t) \geq \zeta(t)$, for a function ζ and t in some compact set $S \subset \mathbb{R}$. Our general result characterizes the MPP in this infinite intersection of events. In the case where the Gaussian process does not have derivatives, the MPP can be expressed as a conditional mean given the points s where it equals $\zeta(s)$. Section 4 gives explicit results for the case $\zeta(t) = t$ and $S = [0, 1]$, the so-called "busy period problem". We illustrate the impact of the smoothness with examples of both a process without derivatives (fBm) and a process with one derivative (integrated Ornstein–Uhlenbeck process). In the case of fBm, we prove that for $H > \frac{1}{2}$ the MPP is *at* the diagonal in some interval $[0, s^*]$, and also at time 1, but *strictly above* the diagonal in between; for $H < \frac{1}{2}$, the corresponding path departs immediately after time 0 from the diagonal, but returns to it strictly before time 1 and continues along it for the rest of the interval. In the case