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Senile reinforced random walks

M. Holmes[a,*], A. Sakai[b]

[a] *Department of Mathematics and Computer Science, T.U. Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands*

[b] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*

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Abstract

We consider random walks with transition probabilities depending on the number of consecutive traversals n of the edge most recently traversed. Such walks may get stuck on a single edge, or have every vertex recurrent or every vertex transient, depending on the reinforcement function $f(n)$ that characterizes the model. We prove recurrence/transience results when the walk does not get stuck on a single edge. We also show that the diffusion constant need not be monotone in the reinforcement.

MSC: 60G50; 60K35

Keywords: Random walk; Reinforcement; Recurrence/transience; Diffusion constant

1. Introduction

Random walks with edge reinforcement were introduced by Coppersmith and Diaconis [3]. Many problems that are simple to state remain unsolved for edge-reinforced random walks on $\mathbb{Z}^d$; however there are also many interesting existing results in the general theory of reinforced random walks. There are strong results for example in one dimension [4], for linear reinforcement on finite graphs [11] and for once-reinforcement on trees [5]. In the case of linear reinforcement there is also an interesting connection with the random walk in a random environment (see for example [16]). The most recent survey that we know of is [17].

* Corresponding author. Tel.: +31 40 2478123; fax: +31 40 2478190.
E-mail address: holmes@eurandom.tue.nl (M. Holmes).

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A nearest-neighbour *senile reinforced random walk* on $\mathbb{Z}^d$, $\{S_n\}_{n\geq 0}$, begins at the origin and initially steps to one of the $2d$ nearest neighbours with equal probability. Subsequent steps are defined in terms of a function $f : \mathbb{N} \mapsto [-1, \infty)$ such that if the current undirected edge $\{S_{n-1}, S_n\}$ has been traversed m consecutive times in the immediate past, then the probability of traversing that edge in the next step is $\frac{1+f(m)}{2d+f(m)}$ with the rest of the possible $2d-1$ choices being equally likely. The reinforcement of the current edge continues until a new edge is traversed, at which point the reinforcement of the previous edge is forgotten (i.e. the weight of that edge returns to its initial value). The special case $f \equiv C$, which we might call a *once-reinforced* senile random walk, or *memory-1 reinforced random walk*, is among the class of walks considered by Gillis [7]. In [7], the natural recurrence and transience results were obtained for $d = 1$ and for d an even integer by generating function analysis. The corresponding results for more general memory-1 models in all dimensions have since been obtained (see for example [2] and references therein). Further extensions to models with memory m have also been studied extensively in the literature (see for example [1,8]), particularly in one dimension. A crucial ingredient in much of the literature is the fact that these models can be described in terms of finite-state Markov chains. For example, for memory-1 models the sequence of increments of the walk is a Markov chain on the space of allowable steps. This is not true for senile random walks in general, although it is implicit in our analysis and explicit in [9] that the senile random walk observed at certain stopping times does have this property.

At the completion of our work we were made aware of two papers [10,15] in which a different model with a similar flavour was studied. Their model has the property that the walk prefers (as defined by the reinforcement function) to continue in the same direction, rather than traverse the same edge, and as such we might call their model a *senile persistent random walk*. Our methods are somewhat different to those used in [10,15], and are much more complicated in the case of the senile persistent random walk. On the other hand, due to the importance of the parity (even/odd) of the number of times an edge is traversed by the senile reinforced random walk, the methods used in [10,15] are not immediately applicable to our model. It should be noted that both the recurrence/transience criteria and the appropriate scaling limits of these two models are not the same in general.

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$ such that $o \notin \mathcal{S}$, $\{y \in \mathbb{Z}^d : |y| = 1\} \subseteq \mathcal{S}$ and $x \in \mathcal{S} \Rightarrow -x \in \mathcal{S}$. We say that there is an edge between $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$ and write $x \sim y$ if $x - y \in \mathcal{S}$. Formally, a senile random walk (SeRW$_{f,\mathcal{S}}$) is a sequence $\{S_n\}_{n\geq 0}$ of $\mathbb{Z}^d$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_f)$ (with corresponding filtration $\{\mathcal{F}_n = \sigma(S_0, \ldots, S_n)\}_{n\geq 0}$) defined by:

- The walk begins at the origin of $\mathbb{Z}^d$, i.e. $S_0 = o$, $\mathbb{P}_f$-almost surely.
- $\mathbb{P}_f(S_1 = x) = D(x)$, where $D(x) = \frac{1}{|\mathcal{S}|}\mathbb{1}_{\{x\in\mathcal{S}\}}$.
- For $n \in \mathbb{N}$, $e_n = \{S_{n-1}, S_n\}$ is a random *undirected* edge ($\mathcal{F}_n$-measurable) and

$$m_n = \max\{k \geq 1 : e_{n-l+1} = e_n \text{ for all } 1 \leq l \leq k\} \tag{1.1}$$

  is an $\mathbb{N}$-valued ($\mathcal{F}_n$-measurable) random variable.
- For $n \in \mathbb{N}$ and $x \in \mathcal{S}$,

$$\mathbb{P}_f(S_{n+1} = S_n + x|\mathcal{F}_n) = \begin{cases} \dfrac{1 + f(m_n)}{|\mathcal{S}| + f(m_n)}, & \text{if } \{S_n, S_n + x\} = e_n, \\ \dfrac{1}{|\mathcal{S}| + f(m_n)}, & \text{if } \{S_n, S_n + x\} \neq e_n. \end{cases} \tag{1.2}$$