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Stochastic Processes and their Applications 124 (2014) 2363–2387

stochastic processes and their applications

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Stochastic integration for tempered fractional Brownian motion

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Received 12 November 2013; received in revised form 5 March 2014; accepted 6 March 2014
Available online 13 March 2014

Abstract

Tempered fractional Brownian motion is obtained when the power law kernel in the moving average representation of a fractional Brownian motion is multiplied by an exponential tempering factor. This paper develops the theory of stochastic integrals for tempered fractional Brownian motion. Along the way, we develop some basic results on tempered fractional calculus.

Keywords: Tempered fractional Brownian motion; Fractional calculus; Stochastic integration; Fractional Sobolev space

1. Introduction

This paper develops the theory of stochastic integration for tempered fractional Brownian motion (TFBM). Our approach follows the seminal work of Pipiras and Taqqu [34] for fractional Brownian motion (FBM). An FBM is the fractional derivative (or integral) of a Brownian motion, in a sense made precise by [34]. A fractional derivative is a (distributional) convolution with a power law [29,32,37]. Recently, some authors have proposed a tempered fractional derivative [2,6] that multiplies the power law kernel by an exponential tempering factor. Tempering produces a more tractable mathematical object, and can be made arbitrarily light, so that the resulting operator approximates the fractional derivative to any desired degree of accuracy over

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http://dx.doi.org/10.1016/j.spa.2014.03.002
0304-4149/

a finite interval. Based on this work, the authors of this paper have recently proposed a tempered fractional Brownian motion (TFBM), see [28] for basic definitions and properties.

Kolmogorov [22] first defined FBM using the harmonizable representation, as a model for turbulence in the inertial range (moderate frequencies). Mandelbrot and Van Ness [26] later developed the moving average representation of FBM. Since then, FBM has found many diverse applications in almost every field of science and engineering [1,12,35]. Davenport [10] modified the power spectrum of FBM to obtain a model for wind speed, which is now widely used [24,31,33]. The authors showed in [28] that TFBM has the Davenport spectrum, and hence TFBM offers a useful extension of the Kolmogorov model for turbulence, to include low frequencies.

The structure of the paper is as follows. In Section 2 we prove some basic results on tempered fractional calculus, which will be needed in the sequel. In Section 3 we apply the methods of Section 2 to construct a suitable theory of stochastic integration for tempered fractional Brownian motion. Finally, in Section 4 we discuss model extensions, related results, and some open questions.

2. Tempered fractional calculus

In this section, we define tempered fractional integrals and derivatives, and establish their essential properties. These results will form the foundation of the stochastic integration theory developed in Section 3. We begin with the definition of a tempered fractional integral.

Definition 2.1. For any $f \in L^p(\mathbb{R})$ (where $1 \leq p < \infty$), the positive and negative tempered fractional integrals are defined by

$$\mathbb{I}_{+}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} f(u)(t-u)_{+}^{\alpha-1} e^{-\lambda(t-u)_{+}} du \tag{2.1}$$

and

$$\mathbb{I}_{-}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} f(u)(u-t)_{+}^{\alpha-1} e^{-\lambda(u-t)_{+}} du \tag{2.2}$$

respectively, for any $\alpha > 0$ and $\lambda > 0$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$ is the Euler gamma function, and $(x)_+ = xI(x > 0)$.

When $\lambda = 0$ these definitions reduce to the (positive and negative) Riemann–Liouville fractional integral [29,32,37], which extends the usual operation of iterated integration to a fractional order. When $\lambda = 1$, the operator (2.1) is called the Bessel fractional integral [37, Section 18.4].

Lemma 2.2. *For any $\alpha > 0$, $\lambda > 0$, and $p \geq 1$, $\mathbb{I}_{\pm}^{\alpha,\lambda}$ is a bounded linear operator on $L^p(\mathbb{R})$ such that*

$$\|\mathbb{I}_{\pm}^{\alpha,\lambda} f\|_p \leq \lambda^{-\alpha} \|f\|_p \tag{2.3}$$

for all $f \in L^p(\mathbb{R})$.

Proof. Young's Theorem [37, p. 12] states that if $\phi \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ then $\phi * f \in L^p(\mathbb{R})$ and the inequality

$$\|\phi * f\|_p \leq \|\phi\|_1 \|f\|_p \tag{2.4}$$