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Stochastic Processes and their Applications 124 (2014) 2415–2441

stochastic processes and their applications

www.elsevier.com/locate/spa

On a class of self-similar processes with stationary increments in higher order Wiener chaoses

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Received 31 July 2013; received in revised form 14 January 2014; accepted 26 February 2014
Available online 6 March 2014

Abstract

We study a class of self-similar processes with stationary increments belonging to higher order Wiener chaoses which are similar to Hermite processes. We obtain an almost sure wavelet-like expansion of these processes. This allows us to compute the pointwise and local Hölder regularity of sample paths and to analyse their behaviour at infinity. We also provide some results on the Hausdorff dimension of the range and graphs of multidimensional anisotropic self-similar processes with stationary increments defined by multiple Wiener–Itô integrals.

MSC: 60 G 18; 60 G 17; 60 G 22; 42 C 40; 28 A 78; 28 A 80

Keywords: Self-similar processes; Multiple Wiener–Itô integral; Wavelet expansion; Hölder regularity; Hausdorff dimension

1. Introduction and background

Self-similar processes with stationary increments (SSSI processes), *i.e.* processes X which satisfy:

$$\forall c > 0 \quad \{X_{ct} : t \in \mathbb{R}_+\} \overset{(d)}{=} \{c^H X_t : t \in \mathbb{R}_+\}$$
$$\forall h > 0 \quad \{X_{t+h} - X_h : t \in \mathbb{R}_+\} \overset{(d)}{=} \{X_t : t \in \mathbb{R}_+\}$$

for some positive H, have been studied for a long time due to their importance both in theory and in practice. Such processes appear as limits in various normalization procedures [18,28,32]. In

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http://dx.doi.org/10.1016/j.spa.2014.02.012
0304-4149/

addition, they are the only possible tangent processes [13]. In applications, they occur in various fields such as hydrology, biomedicine and image processing. The simplest SSSI processes are simply Brownian motion and, more generally, Lévy stable motions. Apart from these cases, the best known such process is probably fractional Brownian motion (fBm), which was introduced in [16] and popularized by [19]. A construction of SSSI processes that generalizes fBm to higher order Wiener chaoses was proposed in [22]. These processes read

$$\forall t \in \mathbb{R}_+ \quad X_t = \int_{\mathbb{R}^d} h_t(x_1, \dots, x_d) dB_{x_1} \dots dB_{x_d}$$

where $\{B_x : x \in \mathbb{R}\}$ is a two-sided Brownian motion and where h_t satisfies:

1. $h_t \in \hat{L}^2(\mathbb{R}^d)$, where $\hat{L}^2(\mathbb{R}^d)$ denotes the space of square-integrable symmetric functions,
2. $\forall c > 0,\ h_{ct}(cx_1, \dots, cx_d) = c^{H-\frac{d}{2}} h_t(x_1, \dots, x_d)$,
3. $\forall \rho \geq 0,\ h_{t+\rho}(x_1, \dots, x_d) - h_t(x_1, \dots, x_d) = h_\rho(x_1 - t, \dots, x_d - t)$.

Properties 2 and 3 ensure self-similarity and stationarity of the increments of the process. Among the kernels satisfying the above properties, [22] considered:

- $\int_0^t \|\mathbf{s}^* - \mathbf{x}\|_2^{H-\frac{d}{2}-1} ds$,
- $\int_0^t \prod_{j=1}^d (s - x_j)_+^{-(\frac{1}{2}+\frac{1-H}{d})} ds$.

where $\mathbf{s}^* = (s, \dots, s)$, $\|\cdot\|_2$ is the euclidean norm in $\mathbb{R}^d$ and $(x)_+ = \max(0, x)$.

The second kernel defines a class of processes called *Hermite processes*, which have been and still are the subject of considerable interest [10,32,1,26,33]. For $d = 1$, one recovers fBm (the only SSSI Gaussian process). The process obtained with $d = 2$ is called the *Rosenblatt* process.

When $d > 1$, the processes in the general class defined in [22] are somewhat difficult to analyse, because they are no longer Gaussian. In recent years, the family of Hermite processes, and specially the Rosenblatt one, has been studied in depth from the point of views of estimation [7], stochastic calculus [33], distributional properties [34], wavelet-like decomposition for $d = 2$ [26], and more. Wavelet-like decompositions, in particular, are useful for investigating the local properties and providing synthesis algorithms.

General results on sample paths properties of ergodic self-similar processes, such as local or pointwise regularity or Hausdorff dimensions were obtained in [30]. These results apply to SSSI processes belonging to higher order Wiener chaoses. Namely, at any $t \in \mathbb{R}_+$, almost surely, the pointwise and local Hölder exponents are equal to H (see Section 4 for a definition). Nevertheless, in the case of fBm, a lot more is known, both in the one and multidimensional cases [25,11,31,3,2]. However, in the non-Gaussian case $d > 1$, there is still room for improvements. For instance, almost sure pointwise Hölder exponent at any point is not known for SSSI processes belonging to Wiener chaoses of order $d \geq 2$. Of course, by stationarity of increments, self-similarity and the finiteness of every moments of X_1, one has:

$$\forall p > 1, \quad \mathbb{E}[|X_t - X_s|^p] = |t - s|^{pH} \mathbb{E}[(X_1)^p].$$

By Kolmogorov's lemma, X has a modification whose sample paths are Hölder-continuous of all exponents smaller than H (we will always work with such a version in the following), but obtaining the exact almost sure pointwise Hölder exponent at any point is more difficult.

The aim of this article is to study a class of SSSI processes obtained by considering a particular kernel h satisfying conditions 1–3 above. The interest of this class is that it allows one to obtain a wavelet-type expansion for all values of d (and not only for $d = 1$ and $d = 2$ as it is the case for Hermite processes). This expansion permits in turn to deduce sharp local regularity