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Mixed boundary value problems of semilinear elliptic PDEs and BSDEs with singular coefficients

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Abstract

In this paper, we prove that there exists a unique weak (Sobolev) solution to the mixed boundary value problem for a general class of semilinear second order elliptic partial differential equations with singular coefficients. Our approach is probabilistic. The theory of Dirichlet forms and backward stochastic differential equations with singular coefficients and infinite horizon plays a crucial role.

Keywords: Dirichlet forms; Quadratic forms; Fukushima's decomposition; Mixed boundary value problem; Backward stochastic differential equations; Reflecting diffusion processes

1. Introduction

In this paper, our aim is to use probabilistic methods to solve the mixed boundary value problem for semilinear second order elliptic partial differential equations (called PDEs) of the

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following form:

$$\begin{cases} Lu(x) = -F(x, u(x), \nabla u(x)) & \text{on } D \\ \dfrac{1}{2}\dfrac{\partial u}{\partial \gamma}(x) - \langle \widehat{B}, n\rangle(x)u(x) = \Phi(x) & \text{on } \partial D. \end{cases} \tag{1.1}$$

The elliptic operator L is given by:

$$\begin{aligned} L &= \frac{1}{2}\nabla \cdot (A\nabla) + B \cdot \nabla - \nabla \cdot (\hat{B}\cdot) + Q \\ &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} B_i(x)\frac{\partial}{\partial x_i} - \mathrm{div}(\hat{B}\cdot) + Q(x) \end{aligned} \tag{1.2}$$

on a d-dimensional smooth bounded Euclidean domain D.

$A(x) = (a_{ij}(x))_{1\le i,j\le d} : R^d \to R^d \otimes R^d$ is a smooth, symmetric matrix-valued function which is uniformly elliptic. That is, there is a constant $\lambda > 1$ such that

$$\frac{1}{\lambda} I_{d\times d} \le A(\cdot) \le \lambda I_{d\times d}. \tag{1.3}$$

Here $B = (B_1, \ldots, B_d)$ and $\hat{B} = (\hat{B}_1, \ldots, \hat{B}_d) : R^d \to R^d$ are Borel measurable functions, which could be singular, and Q is a real-valued Borel measurable function defined on R^d such that,

$$I_D(|B| + |\hat{B}|) \in L^p(D), \quad I_D|Q| \in L^{p_1}(D)$$

for some $p > d$, $p_1 > \frac{d}{2}$.

L is rigorously determined by the following quadratic form:

$$\begin{aligned} \mathcal{Q}(u, v) := (-Lu, v)_{L^2(D)} &= \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_i \int_D B_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ &\quad - \sum_i \int_D \hat{B}_i(x)\frac{\partial v}{\partial x_i}u(x)dx - \int_D Q(x)u(x)v(x)dx. \end{aligned}$$

Details about the operator L can be found in [9,16,21].

The function $F(\cdot, \cdot, \cdot)$ in (1.1) is a nonlinear function defined on $R^d \times R \times R^d$ and $\Phi(x)$ is a bounded measurable function defined on the boundary ∂D and $\gamma = An$, where n denotes the inward normal vector field defined on the boundary ∂D.

To solve the problem (1.1), it turns out that we need to establish the existence and uniqueness of solutions to backward stochastic differential equations (BSDEs) with singular coefficients and infinite horizon, which is of independent interest.

Probabilistic approaches to the boundary value problem of second order differential operators have been adopted by many authors and the earliest work went back as early as 1944 in [12]. There has been a lot of study on the Dirichlet boundary problems (see [1,8,4,6,10,23]). However, there are not many articles on the probabilistic approaches to the Neumann boundary problem.

When $A = I$, $B = 0$ and $\hat{B} = 0$, the following Neumann boundary problem

$$\begin{cases} \dfrac{1}{2}\triangle u(x) + qu(x) = 0 & \text{on } D \\ \dfrac{1}{2}\dfrac{\partial u}{\partial n}(x) = \phi(x) & \text{on } \partial D \end{cases}$$