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Efficient rare-event simulation for perpetuities

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Abstract

We consider perpetuities of the form

$$D = B_1 \exp(Y_1) + B_2 \exp(Y_1 + Y_2) + \cdots,$$

where the Y_j's and B_j's might be i.i.d. or jointly driven by a suitable Markov chain. We assume that the Y_j's satisfy the so-called Cramér condition with associated root $\theta_* \in (0, \infty)$ and that the tails of the B_j's are appropriately behaved so that D is regularly varying with index θ_*. We illustrate by means of an example that the natural state-independent importance sampling estimator obtained by exponentially tilting the Y_j's according to θ_* fails to provide an efficient estimator (in the sense of appropriately controlling the relative mean squared error as the tail probability of interest gets smaller). Then, we construct estimators based on state-dependent importance sampling that are rigorously shown to be efficient.

Keywords: State-dependent importance sampling; Perpetuities; Tail asymptotics; Lyapunov inequalities; Markov chains

1. Introduction

We consider the problem of developing efficient rare-event simulation methodology for computing the tail of a perpetuity (also known as infinite horizon discounted reward). Perpetuities arise in the context of ruin problems with investments and in the study of financial time series such as ARCH-type processes (see for example, [19,26]).

In the sequel we let $X = (X_n : n \geq 0)$ be an irreducible finite state-space Markov chain (see Section 2 for precise definitions). In addition, let $((\xi_n, \eta_n) : n \geq 1)$ be a sequence of i.i.d.

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(independent and identically distributed) two dimensional r.v.'s (random variables) independent of the process X. Given $X_0 = x_0$ and $D_0 = d_0$ the associated (suitably scaled by a parameter $\Delta > 0$) discounted reward at time n takes the form

$$\begin{aligned} D_n(\Delta) = d_0 + \lambda(X_1, \eta_1)\,\Delta \exp(S_1) + \lambda(X_2, \eta_2)\,\Delta \exp(S_2) + \cdots \\ + \lambda(X_n, \eta_n)\,\Delta \exp(S_n) \end{aligned}$$

where the accumulated rate process $(S_k : k \geq 0)$ satisfies

$$S_{k+1} = S_k + \gamma(X_{k+1}, \xi_{k+1}),$$

given an initial value $S_0 = s_0$. In order to make the notation compact, throughout the rest of the paper we shall often omit the explicit dependence of Δ in $D_n(\Delta)$ and we will simply write D_n. We stress that $\Delta > 0$ has been introduced as a scaling parameter which eventually will be sent to zero. Introducing Δ, as we shall see, will be helpful in the development of the state-dependent importance sampling algorithm that we study here.

The functions $(\gamma(x, z) : x \in \mathcal{S}, z \in \mathbb{R})$ and $(\lambda(x, z) : x \in \mathcal{S}, z \in \mathbb{R})$ are deterministic and represent the discount and reward rates respectively. For simplicity we shall assume that $\lambda(\cdot)$ is non-negative. Define

$$\begin{aligned} \phi_{(s_0,d_0,x_0)}(\Delta) &\triangleq P(D_\infty > 1 | S_0 = s_0, D_0 = d_0, X_0 = x_0) \\ &= P(T_\Delta < \infty | S_0 = s_0, D_0 = d_0, X_0 = x_0), \end{aligned} \tag{1}$$

where $T_\Delta = \inf\{n \geq 0 : D_n(\Delta) > 1\}$.

Throughout this paper the distributions of $\lambda(x, \eta_1)$ and $\gamma(x, \xi_1)$ are assumed to be known both analytically and via simulation, as well as the transition probability of the Markov chain X_i. Our main focus on this paper is on the efficient estimation via Monte Carlo simulation of $\phi(\Delta) \triangleq \phi_{(0,0,x_0)}(\Delta)$ as $\Delta \searrow 0$ under the so-called Cramér condition (to be reviewed in Section 2) which in particular implies (see Theorem 1 below)

$$\phi(\Delta) = c_* \Delta^{\theta_*}(1 + o(1)) \tag{2}$$

for a given pair of constants $c_*, \theta_* \in (0, \infty)$. Note that

$$\phi(\Delta) = P\left(\sum_{k=1}^{\infty} \exp(S_k)\,\lambda(X_k, \eta_k) > \frac{1}{\Delta}\right),$$

so Δ corresponds to the inverse of the tail parameter of interest.

Although our results will be obtained for $s_0 = 0 = d_0$, it is convenient to introduce the slightly more general notation in (1) to deal with the analysis of the state-dependent algorithms that we will introduce.

Approximation (2) is consistent with well known results in the literature (e.g. [22]) and it implies a polynomial rate of decay to zero, in $1/\Delta$, for the tail of the distribution of the perpetuity $\sum_{k=1}^{\infty} \exp(S_k)\,\lambda(X_k, \eta_k)$. The construction of our efficient Monte Carlo procedures is based on importance sampling, which is a variance reduction technique popular in rare-event simulation (see, for instance, [4]). It is important to emphasize that, since our algorithms are based on importance sampling, they allow to efficiently estimate conditional expectations of functions of the sample path of $\{D_n\}$ given that $T_\Delta < \infty$. The computational complexity analysis of the estimation of such conditional expectations is relatively straightforward given the analysis of an importance sampling algorithm based on $\phi(\Delta)$ (see for instance the discussion in [1]). Therefore, as