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Consecutive minors for Dyson's Brownian motions

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Abstract

In 1962, Dyson (1962) introduced dynamics in random matrix models, in particular into GUE (also for $\beta = 1$ and 4), by letting the entries evolve according to independent Ornstein–Uhlenbeck processes. Dyson shows the spectral points of the matrix evolve according to non-intersecting Brownian motions. The present paper shows that the interlacing spectra of two consecutive principal minors form a Markov process (diffusion) as well. This diffusion consists of two sets of Dyson non-intersecting Brownian motions, with a specific interaction respecting the interlacing. This is revealed in the form of the generator, the transition probability and the invariant measure, which are provided here; this is done in all cases: $\beta = 1,\ 2,\ 4$. It is also shown that the spectra of three consecutive minors ceases to be Markovian for $\beta = 2,\ 4$.

MSC: primary 60B20; 60G55; secondary 60J65; 60J10

Keywords: Dyson's Brownian motion; Diffusions on spectra of minors; Markov process

1. Introduction

In 1962, Dyson [8] introduced dynamics in random matrix models, in particular into GUE, by letting the entries evolve according to independent Ornstein–Uhlenbeck processes. According to Dyson, the spectral points of the matrix evolve according to non-intersecting Brownian motions.

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The present paper addresses the question whether taking two consecutive principal minors leads to a diffusion on the two interlacing spectra of the minors, taken together. This is so! The diffusion is given by the Dyson diffusion for each of the spectra, augmented with a strong coupling term, which is responsible for a very specific interaction between the two sets of spectral points, to be explained in this paper. However the motion induced on the spectra of three consecutive minors is non-Markovian, for generic initial conditions. A further question: is the motion of two interlacing spectra a determinantal process? We believe this is not the case; but determinantal processes appear upon looking at a different space–time directions. These issues are addressed in another paper by the authors.

During the last few years, the question of interlacing spectra for GUE-minors have come up in many different contexts. In a recent paper, Johansson and Nordenstam [15], based on domino tilings results of Johansson [14], show that domino tilings of Aztec diamonds provide a good discrete model for the consecutive eigenvalues of GUE-minors. In an effort to put some dynamics in the domino tiling model, Nordenstam [23] then shows that the shuffling algorithm for domino tilings is a discrete version of an interlacing of two Dyson Brownian motions, introduced and investigated by Jon Warren [28]; see also [4]. Recently Gorin and Shkolnikov [11] have introduced a new multilevel β-Dyson process, which generalizes Warren's process, for which the Markov property holds for k consecutive spectra. One might have suspected that the Warren process would coincide with the diffusion on the spectra of two consecutive principal minors. They are different!

Non-intersecting paths and interlaced processes (random walks and continuous processes) have been investigated by several authors in many different interesting directions; see e.g. [22,12,14,13,26,16,24,21,6,16,17,2], just to name a few. In particular, in [26,2], partial differential equations were derived for the Dyson process and related processes.

The plan of this paper is the following. We state precisely all the results in Section 2. Some useful matrix equalities are derived in Section 3 which are used in Section 4 to derive transition densities for the various processes considered. Stochastic differential equations are derived in Sections 5 and 6. The fact that the spectra of three consecutive minors are not Markovian for generic initial conditions is demonstrated in Section 7.

There is a companion paper by the same authors aiming at determining the kernel for the point process related to the Dyson Brownian minor process along space-like paths [1].

For RSK, percolation theory and nonintersecting paths, see Chapter 10 and for Laguerre, Jacobi and tridiagonal ensembles, see Chapter 3 in [10]. In the concluding remarks of [7], M. Defosseux mentions, without proof, that the minor process for Hermitian matrices is not Markovian for more than 3 consecutive minors; see also [6]. In [5], it is shown that for a discrete non-commutative analogue of the Dyson Brownian motion (quantum random walk), Markovianess is established for consecutive minors and non-Markovianess for three consecutive minors.

2. The Ornstein–Uhlenbeck process and Dyson's Brownian motion

Consider the space $\mathcal{H}_n^{(\beta)}$ of $n \times n$ matrices B, with entries $B_{k\ell} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ ($\beta = 1, 2, 4$) satisfying the symmetry conditions

$$B_{k\ell} = B_{\ell k}^{*}. \tag{2.1}$$