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Harmonic functions on Walsh's Brownian motion

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Abstract

We examine a variation of two-dimensional Brownian motion introduced by Walsh that can be described as Brownian motion on the spokes of a (rimless) bicycle wheel. We construct the process by randomly assigning angles to excursions of reflecting Brownian motion. Hence, Walsh's Brownian motion behaves like one-dimensional Brownian motion away from the origin, but differently at the origin as it is immediately sent off in random directions. Given the similarity, we characterize harmonic functions as linear functions on the rays satisfying a slope-averaging property. We also classify superharmonic functions as concave functions on the rays satisfying extra conditions.

MSC: 60G99; 31A05

Keywords: Walsh's Brownian motion; Harmonic functions; Superharmonic functions; Excessive functions

1. Introduction

In 1978, J.B. Walsh introduced a variation of two-dimensional Brownian motion [16]. His variation is a generalization of Itô and McKean's skew Brownian motion in two dimensions [8]. First, note that to construct skew Brownian motion, let $\{B_t; t \geq 0\}$ be a one-dimensional Brownian motion and consider the reflecting Brownian motion $R_t = |B_t|$. Itô and McKean take the excursions from 0 of R_t, and independently assign to each a positive or negative sign

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at random. The resulting process is a one-dimensional diffusion that behaves like Brownian motion away from 0, but at 0 the process is "skewed" in that it is more likely to head one direction over the other depending on the probability of assigning a positive or negative sign to a given excursion. Walsh's Brownian motion is constructed in a similar manner; however, instead of independently assigning positive and negative signs at random, angles are assigned to the excursions of reflecting Brownian motion. So Walsh's Brownian motion is a process in the plane that lives on rays emanating from the origin. It behaves like one-dimensional Brownian motion away from the origin, but "has what might be called a *roundhouse singularity* at the origin: when the process enters it, it, like Stephen Leacock's hero, immediately rides off in all directions at once" [16].

Walsh did not prove that this construction produces a diffusion in $\mathbb{R}^2$ when he first introduced it. But since then many constructions have been given using resolvents [11], from the infinitesimal generator [3], and using excursion theory [14]. In 1989, Barlow, Pitman, and Yor provided another construction using semigroups given in [2], which we review in Section 2. Additionally, the process may be constructed in a manner similar to the construction of the cable process in [1] or [15], for example, where Walsh's Brownian motion is considered a cable process in which one of the vertices has uncountably many rays. However, we note that this approach would require some care, as the processes in [1,15] have at most countably many edges at each vertex.

Our interest in this process arose from a paper of Dayanik and Karatzas [4], concerning the optimal stopping problem for one-dimensional diffusions. In this work, the authors provide a simple classification of excessive functions in terms of concave functions. Their results generalize the following theorem of Dynkin and Yushkevich for one-dimensional Brownian motion found in [5].

Theorem 1.1. *A function is excessive for one-dimensional Brownian motion if and only if it is concave.*

In an effort to eventually extend the results of Dayanik and Karatzas to higher dimensional diffusions, we examine the problem for Walsh's Brownian motion. We begin by classifying the harmonic functions. In this case, we take advantage of the similarities between Walsh's Brownian motion and one-dimensional Brownian motion away from 0. For one-dimensional Brownian motion we have the following well-known description of harmonic functions in both the analytic and probabilistic senses.

Theorem 1.2. *Let $\{B_t; t \geq 0\}$ be a one-dimensional Brownian motion. For any real-valued, locally integrable function h the following are equivalent:*

(i) *For every $x \in \mathbb{R}$ and $\epsilon > 0$,*

$$h(x) = \mathbb{E}^x\left[h(B_{\tau_x(\epsilon)})\right],$$

where $\tau_x(\epsilon) = \inf\{t \geq 0 : |B_t - x| = \epsilon\}$.

(ii) *h is linear, i.e., $h(x) = \alpha x + \beta$, for some constants α and β.*

(iii) *$\{h(B_t); t \geq 0\}$ is a continuous martingale.*

Proof. If h satisfies (i), then for every $x \in \mathbb{R}$ and $\epsilon > 0$, we have

$$h(x) = \mathbb{E}^x\left[h(B_{\tau_x(\epsilon)})\right] = \frac{1}{2}h(x-\epsilon) + \frac{1}{2}h(x+\epsilon). \tag{1.1}$$