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On the small-time behaviour of Lévy-type processes

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Abstract

We show some Chung-type lim inf law of the iterated logarithm results at zero for a class of (pure-jump) Feller or Lévy-type processes. This class includes all Lévy processes. The norming function is given in terms of the symbol of the infinitesimal generator of the process. In the Lévy case, the symbol coincides with the characteristic exponent.

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1. Introduction

We study the short-time behaviour of a class of one-dimensional Feller processes $(X_t)_{t\geqslant 0}$. To do so we identify suitable norming functions u, v, w such that the following Chung-type LIL (law of the iterated logarithm) assertions hold $\mathbb{P}^x$-almost surely:

$$\varliminf_{t\to 0} \frac{\sup_{0\leqslant s\leqslant t} |X_s - x|}{u^{-1}(x, t/\log|\log t|)} = C(x), \tag{1}$$

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$$\varlimsup_{t\to 0} \frac{\sup_{0\leqslant s\leqslant t} |X_s - x|}{v(t,x)} = 0 \quad \text{or } = +\infty, \tag{2}$$

$$\varliminf_{t\to 0} \frac{|X_t - x|}{w(t,x)} = \gamma(x) > 0 \quad \text{or } = +\infty. \tag{3}$$

Assertions of this kind are classical for Brownian motion, the corresponding results for Lévy processes are due to Dupuis [6] and Aurzada, Döring and Savov [1]. The class of Feller processes considered in this paper includes Lévy processes and extends the results of these authors. We will characterize the norming functions with the help of the symbol of the infinitesimal generator of the Feller process. In the case of a Lévy process this becomes a rather simple criterion in terms of the characteristic exponent of the process.

Lévy processes. A (real-valued) Lévy process $(X_t)_{t\geqslant 0}$ is a stochastic process with stationary and independent increments and càdlàg (right continuous with finite left limits) sample paths. The transition function is uniquely determined through the characteristic function which is of the following form:

$$\lambda_t(x,\xi) := \mathbb{E}^x e^{i\xi(X_t - x)} = \mathbb{E}^0 e^{i\xi X_t} = e^{-t\psi(\xi)}, \quad t \geqslant 0,\ \xi \in \mathbb{R}.$$

The *characteristic exponent* $\psi : \mathbb{R} \to \mathbb{C}$ is given by the *Lévy–Khintchine formula*

$$\psi(\xi) = il\xi + \tfrac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{iy\xi} + iy\xi \mathbb{1}_{(0,1]}(|y|)\right) \nu(dy) \tag{4}$$

and the *Lévy triplet* (l, σ^2, ν) where ν is a measure on $\mathbb{R}\setminus\{0\}$ such that $\int_{y\neq 0}(1\wedge y^2)\,\nu(dy) < \infty$, and $l \in \mathbb{R}$, $\sigma \geqslant 0$. The characteristic exponent is also the symbol of the infinitesimal generator A of the Lévy process:

$$Au(x) = -\psi(D)u(x) := -\int_{\mathbb{R}} e^{ix\xi}\hat{u}(\xi)\,\psi(\xi)\,d\xi, \quad u \in C_c^\infty(\mathbb{R}),$$

where $\hat{u}(\xi) = (2\pi)^{-1}\int_{\mathbb{R}} u(x)e^{-ix\xi}\,dx$ denotes the Fourier transform of u.

Feller processes. The generator of a Lévy process has *constant coefficients*: it does not depend on the state space variable x. This is due to the fact that a Lévy process is spatially homogeneous which means that the transition semigroup $P_t u(x) = \mathbb{E}^x u(X_t) = \mathbb{E}u(X_t + x)$ is given by convolution operators. We are naturally led to *Feller processes* if we give up spatial homogeneity.

Definition 1. A (one-dimensional) *Feller process* is a real-valued Markov process $(X_t)_{t\geqslant 0}$ whose transition semigroup $P_t u(x) := \mathbb{E}^x u(X_t),\ u \in B_b(\mathbb{R})$, is a *Feller semigroup*, i.e.

(a) P_t is Markovian: if $u \in B_b(\mathbb{R})$, $u \geqslant 0$ then $P_t u \geqslant 0$ and $P_t 1 = 1$;
(b) P_t maps $C_\infty(\mathbb{R}) := \left\{u \in C(\mathbb{R}) : \lim_{|x|\to\infty} u(x) = 0\right\}$ into itself;
(c) P_t is a strongly continuous contraction semigroup in $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$.

Every Lévy process is a Feller process.

Write $(A, D(A))$ for the generator of the Feller semigroup. If $C_c^\infty(\mathbb{R}) \subset D(A)$, then

$$Au(x) = -p(x,D)u(x) := -\int_{\mathbb{R}} e^{ix\xi}\hat{u}(\xi)\,p(x,\xi)\,d\xi, \quad u \in C_c^\infty(\mathbb{R}),$$