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On symmetric and skew Bessel processes☆

Stefan Blei*

Friedrich-Schiller-Universität Jena, Institut für Stochastik, Ernst-Abbe-Platz 2, D-07743 Jena, Germany

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Abstract

We consider the one-dimensional stochastic differential equation

$$X_t = x_0 + B_t + \int_0^t \frac{\delta - 1}{2\, X_s}\, \mathrm{d}s,$$

where $\delta \in (1, 2)$, $x_0 \in \mathbb{R}$ and B is a Brownian motion. For $x_0 \geq 0$, this equation is known to be solved by the δ-dimensional Bessel process and to have many other solutions. The purpose of this paper is to identify the source of non-uniqueness and, from this insight, to transform the equation into a well-posed problem. In fact, we introduce an additional parameter and for each admissible value of this parameter we construct a unique (in law) strong Markov solution of this equation. These solutions are the skew and symmetric Bessel processes, respectively. Moreover, we uncover an alternative way to introduce the δ-dimensional Bessel process.

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Keywords: Bessel process; Bessel equation; Singular stochastic differential equations; Continuous strong Markov processes; Local times; Uniqueness in law; Pathwise uniqueness

1. Introduction

Throughout this paper, $(\Omega, \mathcal{F}, \mathbf{P})$ stands for a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions, i.e., $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$

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* Correspondence to: Friedrich-Schiller-Universität Jena, Fakultät für Mathematik und Informatik, Institut für Stochastik, D-07737 Jena, Germany. Tel.: +49 3641 946261; fax: +49 3641 946252.
E-mail address: stefan.blei@uni-jena.de.

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contains all sets from $\mathcal{F}$ which have **P**-measure zero. For a process $X = (X_t)_{t \geq 0}$ the notation $(X, \mathbb{F})$ indicates that X is $\mathbb{F}$-adapted.

The processes considered in the following belong to the class of continuous semimartingales and local times of continuous semimartingales play an important role in this work. Therefore, in the Appendix we summarize some facts about continuous semimartingales and their local times which we use in the sequel. By writing (A.), we refer to a formula or a result of the Appendix.

In the present paper, we consider the one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + B_t + \int_0^t \frac{\delta - 1}{2\, X_s}\, \mathrm{d}s, \tag{1.1}$$

where $\delta \in (1, 2)$, $x_0 \in \mathbb{R}$ and B is a Wiener process.

A continuous semimartingale $(X, \mathbb{F})$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a (weak) solution of Eq. (1.1) if there exists a Wiener process $(B, \mathbb{F})$ on the same probability space such that Eq. (1.1) holds for all $t \geq 0$ **P**-a.s.

We say that the solution of Eq. (1.1) is unique (in law) if any two solutions $(X^1, \mathbb{F}^1)$ and $(X^2, \mathbb{F}^2)$ with the same initial value defined on the probability spaces $(\Omega^1, \mathcal{F}^1, \mathbf{P}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbf{P}^2)$, respectively, possess the same image law on the space $C_{\mathbb{R}}([0, +\infty))$ of continuous functions defined on $[0, +\infty)$ and taking values in $\mathbb{R}$.

For other equations appearing in the sequel, the notion of a (weak) solution and uniqueness (in law) are defined in the same way.

It is well-known that for $x_0 \geq 0$ Eq. (1.1) is satisfied by a δ-dimensional Bessel process started at x_0 (see e.g. Revuz and Yor [19, Chapter XI, Section 1]). Moreover, as pointed out by Cherny [7, Theorem 3.2], the δ-dimensional Bessel process started at $x_0 \geq 0$ is the unique *non-negative* solution of Eq. (1.1). Of course, the same is true when we allow $\delta \geq 2$. For $\delta \in (0, 1]$, we note that the equation which a δ-dimensional Bessel process solves is of different type than Eq. (1.1) and a Bessel process of dimension $\delta \in (0, 1)$ is not a semimartingale.

Coming back to dimensions $\delta > 1$, there is an important difference between the behaviour of a Bessel process of dimension $\delta \in (1, 2)$ and $\delta \geq 2$. After the Bessel process with dimension $\delta \geq 2$ has started, it never reaches zero. In contrast, the δ-dimensional Bessel process with $\delta \in (1, 2)$ hits zero with probability one and is instantaneously reflected at zero (see [19, Chapter XI, Section 1]). This behaviour of a Bessel process of dimension $\delta \in (1, 2)$ was exploited in [7] to show that for $x_0 \geq 0$ and $\delta \in (1, 2)$ there exist other solutions of Eq. (1.1) than the Bessel process and, therefore, uniqueness does not hold for Eq. (1.1). The author constructed solutions different from the Bessel process by changing the sign of the Bessel process when it reaches zero. In this way, one can obtain a solution of (1.1) that is non-positive after hitting zero. It is also possible to change the sign randomly. When the δ-dimensional Bessel process hits zero one decides by an independent Bernoulli variable whether the sign is changed or not. This kind of construction leads to solutions of (1.1) that are not strong Markov.

In the present article, we identify the source of non-uniqueness of Eq. (1.1). We show by using an invertible space transformation that solving Eq. (1.1) is equivalent to solving an SDE of the following type:

$$Y_t = y_0 + \int_0^t \sigma(Y_s)\, \mathrm{d}B_s + \frac{1}{2}\left(L_+^Y(t, 0) - L_-^Y(t, 0)\right), \tag{1.2}$$

where σ is a certain measurable function and L_+^Y (resp. L_-^Y) stands for the right (resp. left) local time (see Appendix) of the solution process. Similar space transformations, known as Zvonkin