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Hydrodynamic limit for the velocity-flip model

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Abstract

We study the diffusive scaling limit for a chain of N coupled oscillators. In order to provide the system with good ergodic properties, we perturb the Hamiltonian dynamics with random flips of velocities, so that the energy is locally conserved. We derive the hydrodynamic equations by estimating the relative entropy with respect to the local equilibrium state, modified by a correction term.

Keywords: Hydrodynamic limit; Hamiltonian system; Velocity-flip model; Relative entropy method

0. Introduction

This paper aims at proving the hydrodynamic limit for a Hamiltonian system of N coupled oscillators. The ergodic properties of Hamiltonian dynamics are poorly understood, especially when the size of the system goes to infinity. That is why we perturb it by an additional conservative mixing noise, as it has been proposed for the first time by Olla, Varadhan and Yau [16] in the context of gas dynamics, and then in [11] in the context of Hamiltonian lattice dynamics (see e.g. [1,2,7,3,4,6,10,15] for more recent related works).

We are interested in the macroscopic behavior of this system as N goes to infinity, after rescaling space and time with the diffusive scaling. The system is considered under periodic boundary conditions—more precisely we work on the one-dimensional discrete torus $\mathbb{T}_N := \{0, \dots, N-1\}$. The configuration space is denoted by $\Omega_N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$. A typical configuration is given by $\omega = (p_x, r_x)_{x \in \mathbb{T}_N}$ where p_x stands for the velocity of the oscillator at site x, and r_x represents the distance between oscillator x and oscillator $x+1$. The deterministic dynamics is

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described by the harmonic Hamiltonian

$$\mathcal{H}_N = \sum_{x=0}^{N-1} \left[\frac{p_x^2}{2} + \frac{r_x^2}{2} \right]. \tag{0.1}$$

The stochastic perturbation is added only to the velocities, in such a way that the energy of particles is still conserved. Nevertheless, the momentum conservation is no longer valid. The added noise can be easily described: each particle independently waits an exponentially distributed time interval and then flips the sign of velocity. The strength of the noise is regulated by the parameter $\gamma > 0$. The total deformation $\sum r_x$ and the total energy $\sum (p_x^2 + r_x^2)/2$ are the only two conserved quantities. Thus, the Gibbs states are parametrized by two potentials, temperature and tension: for $\beta > 0$ and $\lambda \in \mathbb{R}$, the equilibrium Gibbs measures $\mu^N_{\beta,\lambda}$ on the configuration space $\Omega^N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$ are given by the product measures

$$d\mu^N_{\beta,\lambda} = \prod_{x \in \mathbb{T}_N} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta, \lambda)} dr_x dp_x, \tag{0.2}$$

where $e_x := (p_x^2 + r_x^2)/2$ is the energy of the particle at site x, and $Z(\beta, \lambda)$ is the normalization constant. The temperature is equal to β^{-1} and the tension is given by λ/β.

The goal is to prove that the two empirical profiles associated to the conserved quantities converge in the thermodynamic limit $N \to \infty$ to the macroscopic profiles $\mathbf{r}(t, \cdot)$ and $\mathbf{e}(t, \cdot)$ which satisfy an autonomous system of coupled parabolic equations. Let $\mathbf{r}_0 : \mathbb{T} \to \mathbb{R}$ and $\mathbf{e}_0 : \mathbb{T} \to \mathbb{R}$ be respectively the initial macroscopic deformation profile and the initial macroscopic energy profile defined on the one-dimensional torus $\mathbb{T} = [0, 1]$. We want to show that the functions $\mathbf{r}(t, q)$ and $\mathbf{e}(t, q)$ defined on $\mathbb{R}_+ \times \mathbb{T}$ are solutions of

$$\begin{cases} \partial_t \mathbf{r} = \dfrac{1}{\gamma} \partial_q^2 \mathbf{r}, \\ \partial_t \mathbf{e} = \dfrac{1}{2\gamma} \partial_q^2 \left(\mathbf{e} + \dfrac{\mathbf{r}^2}{2} \right), \end{cases} \qquad q \in \mathbb{T},\ t \in \mathbb{R}, \tag{0.3}$$

with the initial conditions $\mathbf{r}(0, \cdot) = \mathbf{r}_0(\cdot)$ and $\mathbf{e}(0, \cdot) = \mathbf{e}_0(\cdot)$.

We approach this problem by using the relative entropy method, introduced for the first time by H. T. Yau [19] for a gradient[1] diffusive Ginzburg–Landau dynamics. For non-gradient models, Varadhan [18] has proposed an effective approach. Funaki et al. followed his ideas in [12] to extend the relative entropy method to some non-gradient processes and introduced the concept of local equilibrium state of second order approximation.

The usual relative entropy method works with two time-dependent probability measures. Let us denote by μ_0^N the Gibbs local equilibrium associated to a deformation profile $\mathbf{r}_0$ and an energy profile $\mathbf{e}_0$ (see (1.8) for the explicit formula). As we work in the diffusive scaling, we look at the state of the process at time tN^2. We denote it by μ_t^N and we suppose that it starts from μ_0^N. Let $\mu^N_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}$ be the Gibbs local equilibrium associated to the profiles $\mathbf{r}(t, \cdot)$ and $\mathbf{e}(t, \cdot)$ which satisfy (0.3).[2] If we denote by f_t^N and ϕ_t^N, respectively, the densities[3] of μ_t^N and $\mu^N_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}$ with respect

[1] A conservative system is called gradient if the currents corresponding to the conserved quantities are gradients.

[2] For the sake of readability, in the following sections we will denote it by $\mu^N_{\beta_t(\cdot),\lambda_t(\cdot)}$, where $\beta_t(\cdot)$ and $\lambda_t(\cdot)$ are the two potential profiles associated to $\mathbf{r}(t, \cdot)$ and $\mathbf{e}(t, \cdot)$ (see (1.5) and (1.8)).

[3] The existence of these two densities is justified in Section 2.1.