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Time discretization and quantization methods for optimal multiple switching problem☆

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Abstract

In this paper, we study probabilistic numerical methods based on optimal quantization algorithms for computing the solution to optimal multiple switching problems with regime-dependent state process. We first consider a discrete-time approximation of the optimal switching problem, and analyse its rate of convergence. Given a time step h, the error is in general of order $(h\log(1/h))^{1/2}$, and of order $h^{1/2}$ when the switching costs do not depend on the state process. We next propose quantization numerical schemes for the space discretization of the discrete-time Euler state process. A Markovian quantization approach relying on the optimal quantization of the normal distribution arising in the Euler scheme is analysed. In the particular case of uncontrolled state process, we describe an alternative marginal quantization method, which extends the recursive algorithm for optimal stopping problems as in Bally (2003) [1]. A priori L^p-error estimates are stated in terms of quantization errors. Finally, some numerical tests are performed for an optimal switching problem with two regimes.

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1. Introduction

On some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let us introduce the controlled regime-switching diffusion in $\mathbb{R}^d$ governed by

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t,$$

where W is a standard d-dimensional Brownian motion, $\alpha = (\tau_n, \iota_n)_n \in \mathcal{A}$ is the switching control represented by a nondecreasing sequence of stopping times (τ_n) together with a sequence (ι_n) of $\mathcal{F}_{\tau_n}$-measurable random variables valued in a finite set $\{1, \dots, q\}$, and α_t is the current regime process, i.e. $\alpha_t = \iota_n$ for $\tau_n \leq t < \tau_{n+1}$. We then consider the optimal switching problem over a finite horizon:

$$V_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(X_t, \alpha_t)dt + g(X_T, \alpha_T) - \sum_{\tau_n \leq T} c(X_{\tau_n}, \iota_{n-1}, \iota_n)\right]. \tag{1.1}$$

Optimal switching problems can be seen as sequential optimal stopping problems belonging to the class of impulse control problems, and arise in many applied fields, for example in real option pricing in economics and finance. It has attracted a lot of interest during the past decades, and we refer to Chapter 5 in the book [17] and the references therein for a survey of some applications and results in this topic. It is well-known that optimal switching problems are related via the dynamic programming approach to a system of variational inequalities with inter-connected obstacles in the form:

$$\min\left[-\frac{\partial v_i}{\partial t} - b(x, i) \cdot D_x v_i - \frac{1}{2}\mathrm{tr}(\sigma(x, i)\sigma(x, i)' D_x^2 v_i) - f(x, i),\right.$$
$$\left. v_i - \max_{j \neq i}(v_j - c(x, i, j))\right] = 0 \quad \text{on } [0, T) \times \mathbb{R}^d, \tag{1.2}$$

together with the terminal condition $v_i(T, x) = g(x, i)$, for any $i = 1, \dots, q$. Here $v_i(t, x)$ is the value function to the optimal switching problem starting at time $t \in [0, T]$ from the state $X_t = x \in \mathbb{R}^d$ and the regime $\alpha_t = i \in \{1, \dots, q\}$, and the solution to the system (1.2) has to be understood in the weak sense, e.g. viscosity sense.

The purpose of this paper is to solve numerically the optimal switching problem (1.1), and consequently the system of variational inequalities (1.2). These equations can be solved by analytical methods (finite differences, finite elements, etc. ...), see e.g. [14], but are known to require heavy computations, especially in high dimension. Alternatively, when the state process is uncontrolled, i.e. regime-independent, optimal switching problems are connected to multi-dimensional reflected Backward Stochastic Differential Equations (BSDEs) with oblique reflections, as shown in [9,10], and the recent paper [5] introduced a discretely obliquely reflected numerical scheme to solve such BSDEs. From a computational viewpoint, there are rather few papers dealing with numerical experiments for optimal switching problems. The special case of two regimes for switching problems can be reduced to the resolution of a single BSDE with two reflecting barriers when considering the difference value process, and is exploited numerically in [8]. We mention also the paper [4], which solves an optimal switching problem with three regimes by considering a cascade of reflected BSDEs with one reflecting barrier derived from an iteration on the number of switches.

We propose probabilistic numerical methods based on dynamic programming and optimal quantization methods combined with a suitable time discretization procedure for computing the