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Moments, moderate and large deviations for a branching process in a random environment

Chunmao Huang, Quansheng Liu*

LMAM, Université de Bretagne-Sud, Campus de Tohannic, BP 573, 56017 Vannes, France
Université Européenne de Bretagne, France

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Abstract

Let (Z_n) be a supercritical branching process in a random environment ξ, and W be the limit of the normalized population size $Z_n/\mathbb{E}[Z_n|\xi]$. We show large and moderate deviation principles for the sequence $\log Z_n$ (with appropriate normalization). For the proof, we calculate the critical value for the existence of harmonic moments of W, and show an equivalence for all the moments of Z_n. Central limit theorems on $W - W_n$ and $\log Z_n$ are also established.

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1. Introduction and main results

As an important extension of the Galton–Watson process, the model of branching process in a random environment was introduced first by Smith and Wilkinson [22] for the independent environment case, and then by Athreya and Karlin [4] for the stationary and ergodic environment case. See also [3,23,24] for some basic results on the subject. The study of asymptotic properties of a branching process in a random environment has recently received attention; see for example, [1,2,15,5,6,8,7], among others. Here, for a supercritical branching process (Z_n) in a random

* Corresponding author at: LMAM, Université de Bretagne-Sud, Campus de Tohannic, BP 573, 56017 Vannes, France. Tel.: +33 2 9701 7140; fax: +33 2 9701 7175.

E-mail addresses: sasamao02@gmail.com (C. Huang), quansheng.liu@univ-ubs.fr (Q. Liu).

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environment, we shall mainly show asymptotic properties of the moments of Z_n, and prove moderate and large deviation principles for $(\log Z_n)$. In particular, our result on the annealed harmonic moments completes that of Hambly [12] on the quenched harmonic moments, and extends the corresponding theorem of Ney and Vidyashanker [21] for the Galton–Watson process; our moderate and large deviation principles complete the results of Kozlov [15], Bansaye and Berestycki [5], Bansaye and Böinghoff [6] and Böinghoff and Kersting [8] on large deviations.

Let us give a description of the model. Let $\xi = (\xi_0, \xi_1, \xi_2, \dots)$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in some space Θ, whose realization determines a sequence of probability generating functions

$$f_n(s) = f_{\xi_n}(s) = \sum_{i=0}^{\infty} p_i(\xi_n)s^i, \quad s \in [0, 1], \ p_i(\xi_n) \geq 0, \ \sum_{i=0}^{\infty} p_i(\xi_n) = 1. \tag{1.1}$$

A branching process $(Z_n)_{n\geq 0}$ in the random environment ξ can be defined as follows:

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad n \geq 0, \tag{1.2}$$

where given the environment ξ, $X_{n,i}(i = 1, 2, \dots)$ are independent of each other and independent of Z_n, and have the same distribution determined by f_n.

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is given. As usual, $\mathbb{P}_\xi$ is called *quenched law*. The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\xi \otimes \tau$ in the sense that for all measurable and positive function g, we have

$$\int g d\mathbb{P} = \int\int g(\xi, y)d\mathbb{P}_\xi(y)d\tau(\xi),$$

where τ is the law of the environment ξ. The total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of the annealed law $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}_\xi$ (resp. $\mathbb{P}$) will be denoted $\mathbb{E}_\xi$ (resp. $\mathbb{E}$).

For $\xi = (\xi_0, \xi_1, \dots)$ and $n \geq 0$, define

$$m_n(p) = m_n(p, \xi) = \sum_{i=0}^{\infty} i^p p_i(\xi_n) \quad \text{for } p > 0, \tag{1.3}$$

$$m_n = m_n(1), \qquad \Pi_0 = 1 \text{ and } \Pi_n = m_0 \cdots m_{n-1} \text{ for } n \geq 1. \tag{1.4}$$

Then $m_n(p) = \mathbb{E}_\xi X_{n,i}^p$ and $\Pi_n = \mathbb{E}_\xi Z_n$. It is well known that the normalized population size

$$W_n = \frac{Z_n}{\Pi_n}$$

is a nonnegative martingale under $\mathbb{P}_\xi$ (for each ξ) with respect to the filtration $\mathcal{F}_n = \sigma(\xi, X_{k,i}, 0 \leq k \leq n-1, i = 1, 2, \dots)$, so that the limit

$$W = \lim_{n\to\infty} W_n$$

exists almost sure (a.s.) with $\mathbb{E}W \leq 1$. We shall always assume that