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Stochastic Processes and their Applications 121 (2011) 1565–1587
stochastic processes and their applications
www.elsevier.com/locate/spa

A multilevel Monte Carlo algorithm for Lévy-driven stochastic differential equations

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Received 3 August 2009; received in revised form 28 March 2011; accepted 28 March 2011
Available online 9 April 2011

Abstract

This article introduces and analyzes multilevel Monte Carlo schemes for the evaluation of the expectation $\mathbb{E}[f(Y)]$, where $Y = (Y_t)_{t\in[0,1]}$ is a solution of a stochastic differential equation driven by a Lévy process. Upper bounds are provided for the worst case error over the class of all path dependent measurable functions f, which are Lipschitz continuous with respect to the supremum norm. In the case where the Blumenthal–Getoor index of the driving process is smaller than one, one obtains convergence rates of order $1/\sqrt{n}$, when the computational cost n tends to infinity. This rate is optimal up to logarithms in the case where Y is itself a Lévy process. Furthermore, an error estimate for Blumenthal–Getoor indices larger than one is included together with results of numerical experiments.

MSC: primary 60G51; secondary 60H10; 60J75

Keywords: Multilevel Monte Carlo; Numerical integration; Quadrature; Lévy-driven stochastic differential equation

1. Introduction

In this article, we analyze numerical schemes for the evaluation of

$$S(f) := \mathbb{E}[f(Y)],$$

where $Y = (Y_t)_{t\in[0,1]}$ is a solution to a multivariate stochastic differential equation driven by a multidimensional Lévy process, and $f : D[0, 1] \to \mathbb{R}$ is a Borel measurable mapping from the

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0304-4149/$ - see front matter © 2011 Published by Elsevier B.V.
doi:10.1016/j.spa.2011.03.015

Skorokhod space $D[0, 1]$ of $\mathbb{R}^{d_Y}$-valued functions over the time interval $[0, 1]$ that is Lipschitz continuous with respect to the *supremum* norm.

This is a classical problem, which appears for instance in finance, where Y models the risk neutral stock price and f denotes the payoff of a (possibly path dependent) option, and in the past several concepts have been employed for dealing with it. We refer in particular to [13,14], and [10] for an analysis of the Euler scheme for Lévy-driven SDEs.

Recently, Giles [9] introduced the so called *multilevel Monte Carlo method* in the context of stochastic differential equations, and this turned out to be very efficient when Y is a continuous diffusion. Indeed, it can be shown that it is optimal on the Lipschitz class [6], see also [8,4] for further recent results and [11] for a survey and further references.

In this article, we analyze multilevel Monte Carlo algorithms for the computation of $S(f)$ with a focus on *path dependent* f's that are Lipschitz functions on the space $D[0, 1]$ with the supremum norm. In order to gain approximative solutions we decompose the Lévy process in a purely discontinuous Lévy martingale with discontinuities of size smaller than a threshold parameter and a Lévy process with discontinuities larger than the former parameter. The latter process can be efficiently simulated on an arbitrary finite time set and we apply an Euler approximation to get approximative solutions for the stochastic differential equation (see for instance [14] for an analysis of such approximations).

The article is structured as follows. In the next Section 1.1, the main notation as well as the assumptions for the SDE are stated. Furthermore, basic facts concerning the Lévy–Ito-decomposition of a Lévy process are given as a reminder. The actual algorithm, and in particular the coupled Euler schemes, are described in detail in Section 2, while the main result and the right choice of the parameters of the algorithm are postponed to Section 3. Both depend on the behavior of $\int \frac{|x|^2}{h^2} \wedge 1\, \nu(\mathrm{d}x) (h > 0)$ close to zero and on whether the driving Lévy process has a Brownian component.

Let us explain our main findings in terms of the Blumenthal–Getoor index (BG-index) β of the driving Lévy process which is an index in $[0, 2]$. It measures the frequency of small jumps, see (12), where a large index corresponds to a process which has small jumps at high frequencies. If the Blumenthal–Getoor index is smaller than one, appropriately adjusted algorithms achieve the same error bounds as those obtained in Giles [9] for continuous diffusions i.e., the error is of the order $n^{-1/2}(\log n)^{3/2}$ in terms of the computation time n (in the case where f depends on the whole trajectory and is Lipschitz w.r.t. the supremum norm). If the driving Lévy process does not include a Wiener process one even obtains error estimates of order $n^{-1/2}$. Unfortunately the error rates become significantly worse for larger Blumenthal–Getoor indices. In this case, a remedy would be to incorporate a Gaussian term as compensation for the disregarded discontinuous Lévy martingale (see for instance [3]).

Derivations of convergence rates for multilevel schemes are typically based on a *weak* and a *strong* error estimate for the approximative solutions. In this article, the main technical tool is an error estimate in the *strong sense*. We shall use as weak error estimate the one that is induced by the strong estimate. As is well known this approach is suboptimal when the payoff $f(Y)$ actually does not depend on the whole trajectory of the process Y but on the value of Y at a particular deterministic time instance. In that case, an analysis based on the weak estimates of [10] or [16] seems to be favorable.

Unfortunately, in the case where f is path dependent, one does not have better error estimates at hand. To gain better results for large BG-indices it is preferable to incorporate a Gaussian correction. In the case where $f(Y)$ depends only on the value of the process at a given time