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Ergodic BSDEs and related PDEs with Neumann boundary conditions

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Abstract

We study a new class of ergodic backward stochastic differential equations (EBSDEs for short) which is linked with semi-linear Neumann type boundary value problems related to ergodic phenomena. The particularity of these problems is that the ergodic constant appears in Neumann boundary conditions. We study the existence and uniqueness of solutions to EBSDEs and the link with partial differential equations. Then we apply these results to optimal ergodic control problems.

MSC: 60H10; 93E20

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1. Introduction

In this paper we study the following type of (Markovian) backward stochastic differential equation with infinite horizon that we shall call ergodic BSDEs or EBSDEs for short: for all $0 \leqslant t \leqslant T < +\infty$,

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s. \tag{1.1}$$

In this equation $(W_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion and (X^x, K^x) is the solution to the following forward stochastic differential equation reflected in a smooth bounded domain

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$G = \{\phi > 0\}$, starting at x and with values in $\mathbb{R}^d$:

$$\begin{aligned} X_t^x &= x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geqslant 0; \\ K_t^x &= \int_0^t \mathbb{1}_{X_s^x \in \partial G}\mathrm{d}K_s^x, \quad K^x \text{ is non-decreasing.} \end{aligned} \tag{1.2}$$

Our aim is to find a triple (Y, Z, μ), where Y, Z are adapted processes taking values in $\mathbb{R}$ and $\mathbb{R}^{1\times d}$ respectively. $\psi : \mathbb{R}^d \times \mathbb{R}^{1\times d} \to \mathbb{R}$ is a given function. Finally, λ and μ are constants: μ, which is called the "boundary ergodic cost", is part of the unknowns while λ is a given constant.

It is now well known that BSDEs provide an efficient alternative tool to study optimal control problems, see, e.g. [1] or [2]. But to the best of our knowledge, the paper of Fuhrman, Hu and Tessitore [3] is the only one in which BSDE techniques are applied to optimal control problems with ergodic cost functionals that are functionals depending only on the asymptotic behavior of the state (see e.g. costs defined in formulas (1.6) and (1.7) below). This paper deals with the same type of EBSDE as Eq. (1.1) but without boundary condition (and in infinite dimension): their aim is to find a triple (Y, Z, λ) such that for all $0 \leqslant t \leqslant T < +\infty$,

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \tag{1.3}$$

where $(W_t)_{t\geqslant 0}$ is a cylindrical Wiener process in a Hilbert space and X^x is the solution to a forward stochastic differential equation starting at x and with values in a Banach space. In this case, λ is the "ergodic cost".

There is a fairly large amount of literature dealing by analytical techniques with optimal ergodic control problems without boundary conditions for finite-dimensional stochastic state equations. We just mention papers of Arisawa and Lions [4] and Arisawa [5]. In this framework, the problem is treated through the study of the corresponding Hamilton–Jacobi–Bellman equation. Of course, same questions have been studied in bounded (or unbounded) domains with suitable boundary conditions. For example we refer the reader to Bensoussan and Frehse [6] in the case of homogeneous Neumann boundary conditions and to Lasry and Lions [7] for state-constraint boundary conditions. But in all these works, the constant μ does not appear and the authors are interested in the constant λ instead.

To the best of our knowledge, the only works where the problem of the constant μ appears in the boundary condition of a bounded domain are those of Arisawa [8] and Barles and Da Lio [9]. The purpose of the present paper is to show that backward stochastic differential equations are an alternative tool to treat such "boundary ergodic control problems". It is worth pointing out that the role of the two constants are different: our main results say that, for any λ and under appropriate hypothesis, there exists a constant μ for which (1.1) has a solution. At first sight λ does not seem to be important and could be incorporated to ψ, but our proof strategy needs it: we first show that, for any μ, there exists a unique constant $\lambda := \lambda(\mu)$ for which (1.1) has a solution and then we prove that $\lambda(\mathbb{R}) = \mathbb{R}$.

To be more precise, we begin to deal with EBSDEs with zero Neumann boundary condition in a bounded convex smooth domain. As in [3], we introduce the class of strictly monotonic backward stochastic differential equations

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [\psi(X_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha}\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T < +\infty, \tag{1.4}$$