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Isotropic stochastic flow of homeomorphisms on $\mathbb{R}^d$ associated with the critical Sobolev exponent

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Abstract

We consider the critical Sobolev isotropic Brownian flow in $\mathbb{R}^d$ ($d \geq 2$). On the basis of the work of LeJan and Raimond [Y. LeJan, O. Raimond, Integration of Brownian vector fields, Ann. Probab. 30 (2002) 826–873], we prove that the corresponding flow is a flow of homeomorphisms. As an application, we construct an explicit solution, which is also unique in a certain space, to the stochastic transport equation when the associated Gaussian vector fields are divergence free.

MSC: 76F05; 60H10; 76F10

Keywords: Isotropic flow; Stochastic differential equation; Non-Lipschitz condition; Limit theorem; Stochastic transport equation

1. Introduction

Many authors have considered the isotropic stochastic flows, such as Baxendale and Harris [1], Fang and Zhang [6], LeJan and Raimond [9]. A function $C : \mathbb{R}^{d+d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ is called a stationary isotropic covariance function, if $C(x, y) = C(x - y)$ depends only on $x - y$, and for any $d \times d$ real orthogonal matrix G, we have $C(z) = G^{\mathrm{T}} C(Gz)G$, $z \in \mathbb{R}^d$, where G^{T} is the transpose of G. C has the following expression (see also [9]):

$$C^{ij}(z) = \delta^{ij} B_N(|z|) + \frac{z^i z^j}{|z|^2}(B_L(|z|) - B_N(|z|)), \tag{1.1}$$

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with

$$B_L(r) = \int\int \cos(\rho u_1 r) u_1^2 \omega(\mathrm{d}u)\,(F_L(\mathrm{d}\rho) - F_N(\mathrm{d}\rho)) + \int\int \cos(\rho u_1 r)\omega(\mathrm{d}u) F_N(\mathrm{d}\rho), \tag{1.2}$$

$$B_N(r) = \int\int \cos(\rho u_1 r) u_2^2\, \omega(\mathrm{d}u)\,(F_L(\mathrm{d}\rho) - F_N(\mathrm{d}\rho)) + \int\int \cos(\rho u_1 r)\omega(\mathrm{d}u) F_N(\mathrm{d}\rho), \tag{1.3}$$

where $\omega(\mathrm{d}u)$ denotes the normalized Lebesgue measure on S^{d-1}, and F_L, F_N are arbitrary nonnegative measures on $\mathbb{R}_+$. F_L and F_N represent respectively the gradient part and the divergence-free part of the associated Gaussian vector field.

In this paper we consider a particular class of isotropic covariance functions, namely for $\alpha > 0$, let

$$F(\mathrm{d}\rho) = \frac{\rho^{d-1}\mathrm{d}\rho}{(1+\rho^2)^{(d+\alpha)/2}},$$

$$F_L(\mathrm{d}\rho) = aF(\mathrm{d}\rho) \quad \text{and} \quad F_N(\mathrm{d}\rho) = \frac{b}{d-1}F(\mathrm{d}\rho),$$

where $a, b \geq 0$, $a + b > 0$. In the Fourier representation ($c > 0$ is a constant),

$$\hat{C}^{ij}(\xi) = c\left(1 + |\xi|^2\right)^{-(d+\alpha)/2}\left(a\frac{\xi^i\xi^j}{|\xi|^2} + \frac{b}{d-1}\left(\delta^{ij} - \frac{\xi^i\xi^j}{|\xi|^2}\right)\right). \tag{1.4}$$

Let H_C be the reproducing kernel Hilbert space corresponding to the kernel function C, then $H_C \subset H_2^{(d+\alpha)/2}(\mathbb{R}^d, \mathbb{R}^d)$ and (cf. [10])

$$\begin{cases} \text{if } a > 0, b > 0, & H_C = H_2^{(d+\alpha)/2}(\mathbb{R}^d, \mathbb{R}^d), \\ \text{if } a = 0, b > 0, & H_C \text{ is the subset of divergence-free fields in } H_2^{(d+\alpha)/2}(\mathbb{R}^d, \mathbb{R}^d), \\ \text{if } a > 0, b = 0, & H_C \text{ is the subset of gradient fields in } H_2^{(d+\alpha)/2}(\mathbb{R}^d, \mathbb{R}^d). \end{cases} \tag{$*$}$$

Choose a complete orthonormal system $\{A_k : 1 \leq k \leq \infty\}$ of H_C, then

$$\sum_{k=1}^{\infty} A_k^i(x)A_k^j(y) = C^{ij}(x-y), \quad x, y \in \mathbb{R}^d, 1 \leq i, j \leq d. \tag{1.5}$$

In particular, for any $x \in \mathbb{R}^d$, $\sum_{k=1}^{\infty} A_k^i(x)A_k^j(x) = C^{ij}(0)$ is a constant.

Consider the stochastic differential equation (s.d.e in short)

$$\mathrm{d}X_t = \sum_{k=1}^{\infty} A_k(X_t)\mathrm{d}w_t^k, \qquad X_0 = x, \tag{1.6}$$

where $\{w_t^k : 1 \leq k \leq \infty\}$ is a sequence of independent one-dimensional Brownian motions. Generally speaking, this equation does not have a strong solution. When α is small, LeJan and Raimond have introduced in [9] the notion of statistical solutions and shown the phenomenon of phase transition for $0 < \alpha < 2$. It was also shown in [9] that if $\alpha > 2$, the statistical solution