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Stochastic Processes and their Applications 118 (2008) 1264–1277
stochastic processes and their applications
www.elsevier.com/locate/spa

Propagation of singularities in the semi-fractional Brownian sheet

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Received 4 March 2003; received in revised form 30 November 2006; accepted 2 June 2007
Available online 11 September 2007

Abstract

Let X be a semi-fractional Brownian sheet, that is a centred and continuous Gaussian random field with $\mathbb{E}[X(s,t)X(\hat{s},\hat{t})] = (t \wedge \hat{t})(s^\alpha + \hat{s}^\alpha - |s - \hat{s}|^\alpha)/2$. We provide, for $\alpha \in (0, 2)$, an analysis of the propagation of singularities into the fractional direction of X. Here, singularities are times where the law of the iterated logarithm fails, such as fast points.

MSC: primary 60G15; 60G17; secondary 60G60

Keywords: Propagation of singularities; Fractional Brownian motion; Semi-fractional Brownian sheet; Gaussian random field; Fast points

1. Introduction

Let B be a standard Brownian motion on $\mathbb{R}$ and denote by

$$F(\lambda) = \left\{ t \geq 0 : \limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \ln(1/h)}} \geq \lambda \right\}$$

the random set of λ-*fast* or λ-*rapid points*. Orey and Taylor [8] proved that for $0 < \lambda \leq 1$, the set $F(\lambda)$ has almost surely Hausdorff dimension $1 - \lambda^2$. Fast points give rise to a notion of

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2007.06.010

singularities, since for such $t \in F(\lambda)$, one has that the law of the iterated logarithm (LIL) fails, i.e.

$$\limsup_{h\to 0+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\ln\ln(1/h)}} = \infty.$$

We call such a time where the LIL fails a *singularity* (see [9]), and denote the random collection of times which are singularities by $Si(\mathbb{R}_+)$. From Orey and Taylor's dimension formula it follows immediately that $Si(\mathbb{R}_+)$ has Hausdorff dimension 1, almost surely.

In the two-dimensional setting, there are singularities which have a striking property. Fix $\alpha \in (0,2)$ and define X to be a *semi-fractional Brownian sheet*, that is, a centred and continuous Gaussian random field with

$$\mathbb{E}\left[X(s,t)X(\hat{s},\hat{t})\right] = (t\wedge\hat{t})(s^\alpha+\hat{s}^\alpha-|s-\hat{s}|^\alpha)/2,$$

for $(s,t),(\hat{s},\hat{t}) \in (\mathbb{R}_+)^2$. In the s-coordinate X is a *fractional Brownian motion* with Hurst parameter $\alpha/2$, and in the t-coordinate X is a standard Brownian motion. For $\alpha = 1$, the random field X is often called a *Brownian sheet*. Walsh [9] showed that for the Brownian sheet there are singularities which propagate along characteristic directions, i.e. there exists a positive and finite random time T associated properly with some s^* such that

$$\limsup_{h\to 0+} \frac{|X(s^*,T+h)-X(s^*,T)|}{\sqrt{2h\ln\ln(1/h)}} = \infty \tag{1}$$

with probability 1, and, moreover, Eq. (1) implies

$$\limsup_{h\to 0+} \frac{|X(s,T+h)-X(s,T)|}{\sqrt{2h\ln\ln(1/h)}} = \infty$$

simultaneously for all s in some continuous set I, almost surely. The size of I depends on a measurability condition on T and can, for example, be the positive real half-line or an interval. Propagating singularities are like wrinkles in a sheet.

The analysis of the propagation of singularities for a Brownian sheet relies heavily on special properties of this process. The independence of its increments is of particular importance. Passing to more general Gaussian fields, one is forced to find proofs which rely on more general principles. We propose the random field X as a prototype to do that. The increments in the fractional coordinate of the field are no longer independent if $\alpha \neq 1$, but positively correlated for $\alpha > 1$ and negatively correlated for $\alpha < 1$.

Our aim in this article is the analysis of the propagation of singularities for X into the direction of the fractional coordinate. The paper is organized as follows.

In Section 2, we show that a *law of the iterated logarithm* for the t-coordinate holds simultaneously for all $s \geq 0$, i.e. we prove in Theorem 2.8 that

$$\mathbb{P}\left\{\limsup_{h\to 0+} \frac{|X(s,t+h)-X(s,t)|}{\sqrt{2h\ln\ln(1/h)}} = \sqrt{s^\alpha} \quad \text{for all } s \geq 0\right\} = 1$$

for each $t \geq 0$. This is one of the principle tools for proving propagation.

In Section 3, we first construct, by means of a Section Theorem (see Theorem 37, p. 18, [2]), a positive and finite random time T which is a singularity almost surely; see Proposition 3.2. Then we prove in Theorem 3.8 that the singularity at T propagates along the fractional coordinate of X. It is interesting to see that *fast points* either speed up or slow down, according to the value