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Two phase transitions for the contact process on small worlds

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Abstract

In our version of Watts and Strogatz's small world model, space is a d-dimensional torus in which each individual has in addition exactly one long-range neighbor chosen at random from the grid. This modification is natural if one thinks of a town where an individual's interactions at school, at work, or in social situations introduce long-range connections. However, this change dramatically alters the behavior of the contact process, producing two phase transitions. We establish this by relating the small world to an infinite "big world" graph where the contact process behavior is similar to the contact process on a tree. We then consider the contact process on a slightly modified small world model in order to show that its behavior is decidedly different from that of the contact process on a tree.

MSC: 60K35

Keywords: Contact process; Small world; Phase transition; Epidemic

1. Introduction

Small world graphs were first introduced by Watts and Strogatz [23]. In their model, they take a one-dimensional ring lattice and connect all pairs of vertices that are distance m or less. They then "rewire" each edge with probability p by moving one of the ends at random, where the new end is chosen uniformly. This leads to a graph that has small diameter but, in contrast to the Erdös–Renyi model, has a nontrivial density of triangles. These are both properties that they

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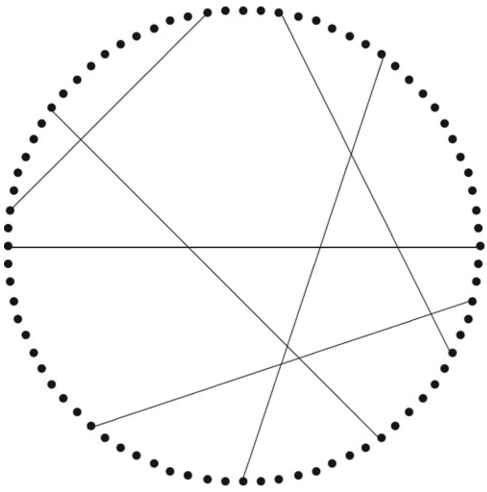

Fig. 1. The Newman and Watts version of a small world graph. For simplicity we have not drawn the short-range connections.

observed in the collaboration graph of film actors, the power grid, and the neural network of the nematode, *C. elegans.*

The small world model has been extensively studied, although most investigators have found it more convenient to study the Newman and Watts [15] version in which all pairs of vertices that are distance m or less are connected (we call these short-range connections), but in addition there is a density p of short-cuts that connect vertices to long-range neighbors chosen at random from the graph. (See Fig. 1). Newman and Watts aren't very precise about what this means but one can, for example if there are n vertices, flip a coin with probability $2p/(n-1)$ of heads to determine if each edge is present.

The graph theoretic properties (e.g., the average distance between two points and the clustering coefficient) of small world graphs are well understood (see [2] for the physicist's view point or [4] for rigorous results). Our focus here will be on the behavior of processes taking place on these networks. Chapter six of Watts [22] discusses the SIR (susceptible–infected–removed) disease model on small world graphs in which individuals that are *susceptible* (state 0) become *infected* (state 1) at a rate proportional to the number of infected neighbors. Infected individuals, after a random amount of time of fixed distribution, become *removed* (state 2), i.e., forever immune to further infection.

The SIR model on the small world graph has a detailed theory due to its connection to percolation: we draw an oriented edge from x to y if x will succeed in infecting y during the time it is infected and the persistence of the epidemic is equivalent to percolation. See Section 8.2 of Newman [14] for what is known about the SIR models on small world graphs. Here we will investigate the more difficult SIS (susceptible–infected–susceptible) model, known to probabilists as the contact process, where recovered individuals are immediately susceptible. Moreno, Pastor-Satorras and Vespignani [11] have studied this model by simulation, but we know of no rigorous results for the contact process on the small world. Berger, Borgs, Chayes and Saberi [5] have proved rigorous results for the contact process on the preferential attachment graph of Barabási and Albert [3]. For more on these results and on random graphs in general see [6].