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Multivariate CARMA processes

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Abstract

A multivariate Lévy-driven continuous time autoregressive moving average (CARMA) model of order (p, q), $q < p$, is introduced. It extends the well-known univariate CARMA and multivariate discrete time ARMA models. We give an explicit construction using a state space representation and a spectral representation of the driving Lévy process. Furthermore, various probabilistic properties of the state space model and the multivariate CARMA process itself are discussed in detail.

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Keywords: CARMA process; Lévy process; Multivariate stochastic differential equation; Spectral representation

1. Introduction

Being the continuous time analogue of the well-known ARMA processes (see e.g. [1]), continuous time ARMA (CARMA) processes, dating back to [2], have been extensively studied over the recent years (see e.g. [3–5] and references therein) and widely used in various areas of application like engineering, finance and the natural sciences (e.g. [6,7,5]). The advantage of continuous time modelling is that it allows handling irregularly spaced time series and in particular high frequency data often appearing in finance. Originally, driving processes of CARMA models were restricted to Brownian motion; however, [4] allowed for Lévy processes which have a finite r-th moment for some $r > 0$.

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As CARMA processes are short memory moving average processes, [8] developed fractionally integrated CARMA (FICARMA) processes, which exhibit long range dependence. So far only univariate CARMA processes have been defined and investigated. However, in order to model the joint behaviour of several time series (e.g. prices of various stocks) multivariate models are required. Thus, we develop multivariate CARMA processes and study their probabilistic properties in this paper.

Unfortunately, it is not straightforward to define multivariate CARMA processes analogously to the univariate ones, as the state space representation (see Section 3.1) relies on the ability to exchange the autoregressive and moving average operators, which is only possible in one dimension. Simply taking this approach would lead to a spectral representation which does not reflect the autoregressive moving average structure. Our approach leads to a model which can be interpreted as a solution to the formal differential equation $P(D)Y(t) = Q(D)DL(t)$, where D denotes the differential operator with respect to t, L a Lévy process and P and Q the autoregressive and moving average polynomial, respectively. Moreover, it is the continuous time analogue of the multivariate ARMA model.

The paper is organized as follows. In Section 2 we review elementary properties of multidimensional Lévy processes and the stochastic integration theory for deterministic functions with respect to them. A brief summary of univariate Lévy-driven CARMA processes forms the first part of the third section and is followed by the development of what will turn out to be the state space representation of multivariate CARMA (MCARMA) processes. We start by constructing a random orthogonal measure allowing for a spectral representation of the driving Lévy process and continue by studying a stochastic differential equation. Analysing the spectral representation of its solution shows that it can be used to define multivariate CARMA processes. After taking a closer look at the probabilistic properties of this SDE (second moments, Markov property, stationary and limiting distributions and path behaviour), we state the definition of MCARMA processes in Section 3.3. Furthermore, we establish a kernel representation, which enables us to derive some further probabilistic properties of MCARMA models. In particular, we characterize the stationary distribution and path behaviour and give conditions for the existence of moments, the existence of a C_b^∞ density as well as for strong mixing.

Throughout this paper we use the following notation. We call the space of all real or complex $m \times m$ matrices $M_m(\mathbb{R})$ or $M_m(\mathbb{C})$, respectively, and the space of all complex invertible $m \times m$ matrices $\mathcal{G}l_m(\mathbb{C})$. Furthermore, A^* denotes the adjoint of the matrix A and $\operatorname{Ker} A$ its kernel. $I_m \in M_m(\mathbb{C})$ is the identity matrix and $\|A\|$ is the operator norm corresponding to the norm $\|x\|$ for $x \in \mathbb{C}^m$. Finally, $I_B(\cdot)$ is the indicator function of the set B and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Multivariate Lévy processes

2.1. Basic facts on multivariate Lévy processes

We state some elementary properties of multivariate Lévy processes that will be needed. For a more general treatment and proofs we refer the reader to [9–11].

We consider a Lévy process $L = \{L(t)\}_{t \geq 0}$ (where $L(0) = 0$ a.s.) in $\mathbb{R}^m$ without a Brownian component determined by its characteristic function in the Lévy–Khintchine form $E\left[e^{i\langle u, L(t)\rangle}\right] = \exp\{t\psi_L(u)\}$, $t \geq 0$, where

$$\psi_L(u) = i\langle \gamma, u\rangle + \int_{\mathbb{R}^m} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle I_{\{\|x\|\leq 1\}})\, \nu(dx), \quad u \in \mathbb{R}^m, \tag{2.1}$$