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Backward stochastic differential equations with singular terminal condition

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Abstract

In this paper, we are concerned with backward stochastic differential equations (BSDE for short) of the following type:

$$Y_t = \xi - \int_t^T Y_r |Y_r|^q \, \mathrm{d}r - \int_t^T Z_r \, \mathrm{d}B_r,$$

where q is a positive constant and ξ is a random variable such that $\mathbb{P}(\xi = +\infty) > 0$. We study the link between these BSDE and the associated Cauchy problem with terminal data g, where $g = +\infty$ on a set of positive Lebesgue measure.

Keywords: Backward stochastic differential equation; Non-integrable data; Viscosity solutions of partial differential equations

Contents

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0. Introduction and main results

Backward stochastic differential equations (BSDE for short in the remainder) are equations of the following type:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dB_r, \quad 0 \leq t \leq T,$$

where $(B_t)_{0 \leq t \leq T}$ is a standard d-dimensional Brownian motion on a probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right)$, with $(\mathcal{F}_t)_{0 \leq t \leq T}$ the standard Brownian filtration. The function $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n$ is called the *generator*, T the *terminal time*, and the $\mathbb{R}^n$-valued $\mathcal{F}_T$-adapted random variable ξ a *terminal condition*.

The unknowns are the processes $\{Y_t\}_{t \in [0,T]}$ and $\{Z_t\}_{t \in [0,T]}$, which are required to be adapted with respect to the filtration of the Brownian motion: this is a crucial point.

Such equations, in the non-linear case, were introduced by Pardoux and Peng in 1990 in [1]. They gave the first existence and uniqueness result for n-dimensional BSDE under the following assumptions: f is Lipschitz continuous in both variables y and z and the data, ξ, and the process, $\{f(t, 0, 0)\}_{t \in [0,T]}$, are square integrable. Since then, BSDE have been studied with great interest. In particular, many efforts have been made to relax the assumptions on the generator and the terminal condition. For instance Briand et al. in [2] proved an existence and uniqueness result under the following assumptions: f is Lipschitz in z, continuous and monotone in y, the data, ξ, and the process, $\{f(t, 0, 0)\}_{t \in [0,T]}$, are in L^p for $p > 1$. The result is still true for $p = 1$ with another technical condition.

The results of [2] are the starting point of this work, where we consider a one-dimensional BSDE with a non-linear generator:

$$Y_t = \xi - \int_t^T Y_r |Y_r|^q\, dr - \int_t^T Z_r\, dB_r \quad \text{with } q \in \mathbb{R}_+^*. \tag{1}$$

The generator $f(y) = -y|y|^q$ satisfies all the assumptions of the Theorems 4.2 and 6.2–6.3 of [2]: f is continuous on $\mathbb{R}$, does not depend on z, and is monotone:

$$\forall (y, y') \in \mathbb{R}^2, \quad (y - y')(f(y) - f(y')) \leq 0. \tag{2}$$