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The origins of Euler's early work on continued fractions

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Abstract

In this paper, I examine Euler's early work on the elementary properties of continued fractions in the 1730s, and investigate its possible links to previous writings on continued fractions by authors such as William Brouncker. By analysing the content of Euler's first paper on continued fractions, 'De fractionibus continuis dissertatio' (1737, published 1744) I conclude that, contrary to what one might expect, Euler's work on continued fractions initially arose not from earlier writings on continued fractions, but from a wish to solve the Riccati differential equation.

Résumé

Dans cet article, j'éxamine les premiers travaux d'Euler sur les propriétés élementaires des fractions continues dans les années 1730, et j'étudie leurs liens possibles avec les travaux précédents sur les fractions continues par auteurs comme William Brouncker. En analysant le contenu du premier article d'Euler sur les fractions continues, 'De fractionibus continuis dissertatio' (1737, publié 1744), je déduis que, contrairement aux attentes, les travaux d'Euler sur les fractions continues n'ont pas fait suit initialement aux travaux plus anciens sur les fractions continues, mais à un souhait de résolver l'équation différentiel de Riccati.

MSC: 01A50; 34A03; 01A45; 26A03

Keywords: Euler; Continued fractions; Riccati equation; 18th-century; Bernoulli; Mathematics

1. Introduction

With the exception of a few isolated results which appeared in the sixteenth and seventeenth centuries, most of the elementary theory of continued fractions was developed in a single paper written in 1737 by Leonhard Euler. In this paper, 'De fractionibus continuis dissertatio' ('Essay on continued fractions',[1] E71 in Gustav Eneström's 1913 index of Euler's works), Euler presented continued fractions as an alternative

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[1] The paper, as with almost all of Euler's published works, is available from http://www.eulerarchive.org/. There also exists a complete English translation of E71: see Wyman and Wyman (1985). However, the translation given there does contain a few misprints and so all translations from Latin or French in the present paper are my own.

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to infinite series or products for representing irrational and transcendental quantities.[2] He established most of the basic properties of continued fractions, briefly examined certain special cases, and then used the properties of continued fractions to arrive at the result for which the paper is arguably best known: namely, the first known proof that the regular continued fraction expansion of e continues infinitely. It follows from this fact (though Euler does not explicitly say so) that e is irrational.[3]

It is not immediately clear why Euler wrote E71, and existing commentaries on the paper have not fully explored this puzzle (Sandifer, 2007a, 2007b; Baltus, 2007). It is known that Euler was well-acquainted with John Wallis's 1656 book *Arithmetica infinitorum* (Knobloch, 1989, 279; Calinger, 1996, 124). The *Arithmetica* contains some of the earliest results in the theory of continued fractions, obtained by William Brouncker and published by Wallis as part of their efforts to find exact expressions for $4/\pi$. At first glance it might, therefore, be tempting to look to the *Arithmetica* as the source of Euler's interest in the subject of continued fractions. However, though Euler clearly had the *Arithmetica infinitorum* in mind when writing E71, the focus of his own paper is completely different. Apart from a brief mention of Wallis and Brouncker at the beginning of the paper, along with a note that Brouncker's derivation of his continued fraction representation of $4/\pi$ remained lost, there are seemingly no connections between the *Arithmetica infinitorum* and E71. Euler did eventually make several attempts to recover Brouncker's missing proof, but the first of these did not occur until 1739, in 'De fractionibus continuis observationes' ('Observations on continued fractions', E123), Euler's second paper on continued fractions.

For this reason, we must look elsewhere for the stimulus or stimuli that drove Euler to write E71. In this paper, I will propose that this stimulus came not from the work of Wallis and Brouncker, but from some early work of Daniel Bernoulli on ordinary differential equations, via some of the letters Euler exchanged with Christian Goldbach in the early 1730s. To do this, I will first outline some seventeenth century results in the basic theory of continued fractions, including work by Rafael Bombelli, Pietro Antonio Cataldi, Daniel Schwenter, John Pell, John Wallis, and William Brouncker. Then, I will briefly explain the content of E71, showing that it has little in common with any of these earlier works. Finally, I will discuss some of the contents of Daniel Bernoulli's 1724 book *Exercitationes quaedam mathematicae*, and a letter which Euler wrote to Christian Goldbach on 25 November 1731, and argue that in fact it was these writings which prompted the main result of E71.

2. Earlier works on continued fractions

2.1. *Bombelli's* L'algebra *(1572) and Cataldi's* Trattato del modo brevissimo *(1613)*

One of the earliest instances of a continued fraction-related method in western mathematics is in the work of Rafael Bombelli. His *L'algebra parte maggiore dell'aritmetica* was published in Bologna in 1572; a second edition followed in 1579. The text is best known now for its treatment of complex numbers. However, it also includes an approximation method for square roots which produces what we now interpret as a continued fraction.[4] It consists of finding an integer approximation to the square root, and then finding values which alternately over- and under-approximate the non-integer part, which when written out in full form a continued fraction. Though Bombelli did not explicitly write out the continued fraction (and, indeed,

[2] The notion of an 'irrational' quantity as used by Euler is equivalent to the modern notion: an irrational quantity is one which cannot be written as a quotient of two whole numbers (Euler, 1771, 54–55). However, his notion of 'transcendental' quantities is somewhat different to the modern sense of the word: see Petrie (2012).

[3] As will be discussed below, the denominators of the regular continued fraction expansion of e form an interpolated arithmetic progression. This was known to Roger Cotes in 1714; however, he only observed that the progression seemed to continue indefinitely, and did not prove that it actually did so (Cotes, 1714, 11). For more details, see Fowler (1999, Chapter 9).

[4] Bombelli's exposition of this method is reproduced in Smith (1959, 80–82).