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ScienceDirect
Historia Mathematica 43 (2016) 133–147
HISTORIA MATHEMATICA
www.elsevier.com/locate/yhmat

Jost Bürgi's method for calculating sines

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Available online 6 April 2016

Abstract

For a long time it has been known that in the 16th century the Swiss mathematician Jost Bürgi found a new method for calculating sines, but no information about the details has been available. Recently a manuscript written by Bürgi himself has come to light in which he explains his algorithm. It is totally different from the traditional procedure which was used until the 17th century. In the first part of the article the standard method is explained which was rooted in Greek antiquity with Ptolemy's computation of chords and which was used in the Arabic-Islamic tradition and in the Western European Middle Ages for calculating chords as well as sines. The main part of the article deals with Bürgi's way. By only using additions and halving, his procedure is elementary and it converges quickly. Bürgi does not explain why his method is correct, but in the last part of the article a modern proof for the correctness of his algorithm is provided.

Zusammenfassung

Schon seit langem ist bekannt, dass im 16. Jahrhundert der Schweizer Mathematiker Jost Bürgi eine neue Methode gefunden hat, um die Sinus-Werte zu berechnen, aber über die Einzelheiten gab es keine Informationen. Jetzt ist eine Handschrift aufgetaucht, in der Bürgi selbst dieses Verfahren beschreibt. Es unterscheidet sich grundsätzlich von dem üblichen Vorgehen, das man bis zum 17. Jahrhundert benutzte. Im ersten Teil des Aufsatzes wird die herkömmliche Methode beschrieben, die auf die Sehnenberechnung des Ptolemaeus in der griechischen Antike zurückgeht und die auch im arabisch-islamischen Bereich und im westeuropäischen Mittelalter benutzt wurde, um Sehnen und Sinuswerte zu berechnen. Im Mittelpunkt des Aufsatzes steht die Erklärung von Bürgis Verfahren. Dadurch, dass er nur Additionen und Halbierungen benutzt, ist seine Methode elementar, und sie konvergiert schnell. Bürgi erklärt nicht, warum sein Verfahren funktioniert. Im letzten Teil des Aufsatzes wird ein moderner Beweis für die Richtigkeit von Bürgis Methode gegeben.

MSC: 01A20; 01A30; 01A35; 01A40; 33B10; 51-03; 97G60
Keywords: Bürgi; Sine calculation; Trigonometrical tables

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http://dx.doi.org/10.1016/j.hm.2016.03.001
0315-0860/

From various sources we know that the Swiss instrument maker and mathematician Jost Bürgi (1552–1632) found a new way of calculating any sine value. Many mathematicians and historians of mathematics have tried to reconstruct his so-called "Kunstweg" ("skillful/artful method"), but they did not succeed. Now a manuscript by Bürgi himself has been found which enables us to understand his procedure. The main purpose of this article is to explain Bürgi's method. It is totally different from the conventional way to calculate sine values which was used until the 17th century. First we will give a brief overview of the early history of trigonometry and the traditional methods for calculating sines.

1. Historical remarks on trigonometry and on trigonometrical tables

The main purpose of the following remarks is to show how values of trigonometric functions, especially of chords and sines, were calculated before the time of Bürgi. For this reason it is necessary to go back to Greek antiquity, the Arabic-Islamic tradition and the Western European Middle Ages. Of course this is not the place to present an extensive history of trigonometry.[1]

In Greek antiquity, trigonometric functions were used in different contexts in order to calculate triangles and quadrangles: in geodesy particularly for determining heights and distances and in astronomy for calculating spherical triangles. In geodesy the main method was to use the proportion of the catheti in plane orthogonal triangles, i.e., in modern terms, the tangent. In astronomy the central problem was to find relations between the chords and the radius of the circle, today expressed by the sine.

When simple geodetic measurements had to be carried out, properties of similar triangles were used to find the fourth proportional with the help of the theorem of intersecting lines. For measuring angular distances on earth or in the sky the Jacob's staff was available.

Astronomical calculations were much more complicated. The main purpose was to solve problems of spherical geometry, i.e. to calculate spherical triangles and quadrangles. The Greeks had developed a method to calculate chords which was based on the fact that the central angle α of a circle is a measure for the circular arc (arc α) and that there is a unique relation between the arc of a circle and the corresponding chord (crd α). Accordingly they developed tables of chords as well as theorems on the relations between chords. In its fully developed form, this system is found in the *Almagest*, the central astronomical work of Ptolemy (1st half of the 2nd c. AD). It was based upon the earlier achievements of Hipparchus (ca. 150 BC).[2]

The calculation of chords and its application to astronomical problems are taught in the first book of the Almagest. In Chapter 11 Ptolemy presents his table of chords. This is the oldest extant table of this kind. For any arc between 0° and 180° at intervals of half a degree it gives the value of the corresponding chord. The arcs and the chords are expressed in the sexagesimal system and the radius, too, is divided sexagesimally.

In chapter 10 Ptolemy explains in detail how he found these values.[3] He proceeds as follows:

1. He starts with chords which correspond to the sides of special regular polygons. From Euclid's *Elements* it was known that regular polygons with sides $n = 3, 4, 5, 6$ and 10 can be constructed easily. The sides of these polygons are the chords of the angles 120°, 90°, 72°, 60° and 36°, accordingly. By theorems given in Euclid's *Elements* Ptolemy calculates the chords of these five angles.
2. Then Ptolemy shows how to find the chord of the sum and difference of two arcs whose chords are known. His proof is based upon the theorem of the cyclic quadrilateral (the so-called "Ptolemy's Theorem").

[1] This can be found in [Van Brummelen, 2009].
[2] On Hipparchus, especially on his table of chords, see [Van Brummelen, 2009, 34–46].
[3] See [Van Brummelen, 2009, 70–77].