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“Geometrical equations”: Forgotten premises of Felix Klein’s *Erlanger Programm*

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Abstract

Felix Klein’s *Erlanger Programm* (1872) has been extensively studied by historians. If the early geometrical works in Klein’s career are now well-known, his links to the theory of algebraic equations before 1872 remain only evoked in the historiography. The aim of this paper is precisely to study this algebraic background, centered around particular equations arising from geometry, and participating on the elaboration of the *Erlanger Programm*. Another result of the investigation is to complete the historiography of algebraic equations, in which those “geometrical equations” do not appear.

Résumé

Le *Programme d’Erlangen* de Felix Klein (1872) a été abondamment étudié par les historiens. Si les premiers travaux géométriques de la carrière de Klein sont maintenant bien connus, ses rapports à la théorie des équations algébriques antérieurs à 1872 ne sont qu’évoqués dans l’historiographie. Le but de cet article est justement d’étudier ce contexte algébrique, centré autour d’équations algébriques particulières provenant de la géométrie, qui participe à l’élaboration du *Programme d’Erlangen*. Un autre résultat de ce travail est la complétion de l’historiographie des équations algébriques, dans laquelle ces “équations de la géométrie” n’apparaissent pas.

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1. Introduction

Felix Klein's *Erlanger Programm*[1] is one of those famous mathematical works that are often invoked by current mathematicians, most of the time in a condensed and modernized form: doing geometry comes down to studying (transformation) group actions on various sets, special attention being paid to invariant objects.[2] As rough and anachronistic this description may be, it does represent Klein's principle to praise the importance of transformation groups and their invariants to unify and classify geometries.

The *Erlanger Programm* has been abundantly studied by historians.[3] In particular, (Hawkins, 1984) questioned the commonly believed revolutionary character of the *Programm*, giving evidence of both a relative ignorance of it by mathematicians from its date of publication (1872) to 1890 and the importance of Sophus Lie's ideas[4] about continuous transformations for its elaboration. These ideas are only a part of the many facets of the *Programm*'s geometrical background of which other parts (non-Euclidean geometries, line geometry) have been studied in detail in (Rowe, 1989b).

Beside this geometrical background, the historiography always mentions Camille Jordan, sometimes for his research on groups of movements, more often for his *Traité des substitutions et des équations algébriques*, (Jordan, 1870b): Klein had been inspired by the presentation given in the *Traité* of a group-theoretic framework for the theory of algebraic equations and sought an analogous transformation group framework for geometry.[5] Whereas the historiography stops there on this algebraic point, this paper will carefully investigate the connections between Klein and the theory of algebraic equations in the period preceding the publication of the *Erlanger Programm*. The purpose of our investigation is obviously not to dismiss the historical works previously cited but rather to help give a more complete and accurate picture of Klein's pre-*Programm* mathematical background. We will argue that Jordan's theory of substitutions as expressed in the chapter of his *Traité* devoted to geometrical applications[6] was actually one part of a broader algebraic background for Klein, a background having "geometrical equations"[7] as core objects. As we will see, an interesting feature of these equations was that they linked algebra and geometry in a way somewhat analogous and yet prior to the way the groups of transformations of the *Programm* did.

One obvious and explicit trace of the theory of algebraic equations or of the theory of substitutions in the *Programm* could be found in the concluding remarks, confirming Klein's desire for an analogy between geometry and the theory of algebraic equations:

> The further problems which we wished to mention arise on comparing the views here set forth with the so-called *Galois* theory of equations. [...]

1 (Klein, 1872), translated to English in (Klein, 1893) and to French in (Klein, 1974).

2 However, this part about invariant objects is often neglected, if ever mentioned. (Perrin, 2002) points out the importance of invariants in the *Erlanger Programm* and suggests a reinforcement of the incorporation of geometrical invariants in the French teaching of mathematics.

3 Beside the references given above and below, see (Gray, 1992, 2005; Rowe, 1983, 1985, 1992).

4 About Lie's early works, see (Rowe, 1989b) and the first chapter of (Hawkins, 2000).

5 For mentions of groups of movements, see (Rowe, 1989b, 211) or Jean Dieudonné in the preface of the French translation of the *Programm*, (Klein, 1974, x). For the *Traité des substitutions et des équations algébriques*, see for instance (Hawkins, 1984, 444) or (Birkhoff and Bennett, 1988, 151). (Wussing, 1969, 132–143) emphasized the existence of group-theoretic links between the *Programm* and the *Traité des substitutions* and showed how the ideas of the *Programm* contributed to the development of group theory; we will here examine the *Traité* and other texts that Wussing does not mention. We will thus reveal more than a simple analogy between groups of substitutions and groups of transformations. (Rowe, 1989b, 211) also discusses the possibility that Klein and Lie actually learned substitution theory from Jordan and his *Traité* while they were in Paris in 1870.

6 Note that this chapter only represented one of the various developments existing in the *Traité*. In particular, Klein's reading of Jordan's book did not seem to focus on the parts devoted to linear groups and general considerations on solubility of equations. See (Brechenmacher, 2011). I am indebted to one of the reviewers for this point.

7 See the end of this introduction for explanations about that expression.