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The mystery of Riemann's curvature

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Abstract

Bernhard Riemann's habilitation lecture of 1854 on the foundations of geometry contains a stunningly precise concept of curvature without any supporting calculations. Another memoir of 1861 contains formulas in which we may recognize our Riemann tensor, though in a different context and without much geometrical interpretation. The first text is mysterious by the lack of formulas, the second by the excess of formulas. The purpose of this essay is to investigate this double mystery and the stimulating effect it had on some of Riemann's early readers, from Richard Dedekind to Tullio Levi-Civita. Use is made of some heretofore unexploited manuscript sheets by Riemann.

Résumé

La leçon d'habilitation que Bernhard Riemann prononça en 1854 sur les fondements de la géométrie contient un concept de courbure étonnamment précis, sans aucun calcul pour le soutenir. Un autre mémoire de 1861 contient des formules parmi lesquelles le lecteur moderne reconnaît le tenseur de Riemann, cela dans un contexte différent et presque sans interprétation géométrique. Le premier texte est mystérieux par le manque de formules, le second par l'excès de formules. Le but de cet essai est d'examiner ce double mystère et l'effet stimulant qu'il eut sur quelques lecteurs de Riemann, de Richard Dedekind à Tullio Levi-Civita. Quelques notes manuscrites inédites de Riemann sont utilisées à cet effet.

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> I still have vivid memories of the extraordinary impression that Riemann's trains of thought made on young mathematicians [when his habilitation lecture was published]. Much of it seemed obscure and hard to understand and yet of unfathomable depth. Today's mathematicians, who have integrated all these things in their way of thinking, still admire the clarity and fertility of the analysis.[1]
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> [Felix Klein, 1926–1927, 2: 16]

0. Introduction

When, in his habilitation lecture of 1854, Bernhard Riemann characterized curvature in a metric manifold of any dimension, he left much for mathematicians and historians to wonder about. The concept there occurred with no calculations whatsoever and yet with stunning precision. Riemann was addressing a mostly non-mathematical audience, the Göttingen faculty of philosophy, who would not have followed the mathematical technicalities. Old Carl Friedrich Gauss probably was the only listener to understand the extraordinary profundity of the lecture. Riemann never published his reflections nor any relevant calculation, because poor health prevented him to do so (according to Dedekind) or maybe because his interest in geometry was only occasional.[2]

In 1861, Riemann sent to the French Academy of sciences a memoir usually called the *Commentatio*, in which the modern reader easily recognizes the precise expression of what is now called the Riemann tensor as well as an argument for the covariance of this tensor. It is tempting to relate the relevant section of this memoir to the geometric problem of finding a criterion of flatness for a manifold, in the wake of the habilitation lecture. Yet the context of the *Commentatio* was different: it answered a prize question on the propagation of heat in solid bodies. Its derivations were purely analytic, and its only contact with geometry was in Riemann's remark that some analytical expression could be illustrated as the curvature components of a metric manifold. The prize was not attributed, and Riemann's submission rested in the archive of the French Academy.[3]

Riemann's habilitation lecture and the *Commentatio* were published posthumously, the former by Richard Dedekind in 1867, and the latter by Heinrich Weber in 1876 in Riemann's *Werke*. Riemann was no longer there to answer the following questions: How did he obtain the results enunciated in the habilitation lecture? Surely he must have done some precise calculations, but what were they? Do the calculations of the *Commentatio* have anything to do with this lecture? Or could Riemann have conceived them in a purely algebraic or analytic manner?

In the absence of relevant manuscript materials, the answer to these questions can only be conjectural. The best we can do is to examine the resources available to Riemann and to guess how he might have exploited them. Most important in this regard are Gauss's *Disquisitiones* of 1828 on curved surfaces, for Riemann himself regarded his concept of curvature as a generalization of the intrinsic curvature invented by Gauss in the two-dimensional case. This is why the first section of this essay is devoted to Gauss's theory of surfaces and to a rephrasing of his main results and proofs in a manner that lends itself to higher-dimensional generalization. I have thereby tried to avoid notions that would not have been available to Riemann.

In the second section, I use this Gaussian background to analyze Riemann's results regarding the curvature of a manifold in his habilitation lecture on the one hand, and his results regarding the transformation properties of quadratic differential forms in the *Commentatio* on the other hand (a *quadratic differential*

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[1] "So habe ich noch lebhafte Erinnerung an den ausserordentlichen Eindruck, den Riemanns Gedankengänge damals auf die jungen Mathematiker machten. Vieles erschien uns dunkel und schwerverständlich und doch wieder von unergründlicher Tiefe, wo der heutige Mathematiker, der alle diese Dinge von vornherein in seine Denkweise aufgenommen hat, nur noch die Klarheit und Prägnanz der Auseinandersetzung bewundert."

[2] On these circumstances, cf. Dedekind (1876, 517).

[3] On the purpose and fate of the *Commentatio*, cf. Farwell and Knee (1990).