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Practicing algebra in late antiquity: The problem-solving of Diophantus of Alexandria

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Abstract

Medieval algebra is distinguished from other arithmetical problem-solving techniques by its structure and technical vocabulary. In an algebraic solution one or several unknowns are named, and via operations on the unknowns the problem is transferred to the artificial setting of an equation expressed in terms of the named powers, which is then simplified and solved. In this article we examine Diophantus' *Arithmetica* from this perspective. We find that indeed Diophantus' method matches medieval algebra in both vocabulary and structure. Just as we see in medieval Arabic and Italian algebra, Diophantus worked out the operations expressed in the enunciation of a problem prior to setting up a polynomial equation. Further, his polynomials were regarded as aggregations with no operations present.

Résumé

L'algèbre médiévale se distingue des autres techniques arithmétiques de résolution de problèmes par sa structure et son vocabulaire spécifique. Dans une solution algébrique on *nomme* une ou plusieurs inconnues, et au moyen d'opérations sur les inconnues le problème est transféré dans le cadre artificiel d'une *équation* exprimée en termes de puissances nommées, qui est alors simplifiée et résolue. C'est sous cette perspective que nous examinons les *Arithmétiques* de Diophante dans le présent article. Ainsi, nous parvenons à la conclusion que la méthode résolutoire de Diophante correspond à l'algèbre médiévale aussi bien au regard du vocabulaire que de la structure. Tout comme dans l'algèbre arabe et italienne médiévale, Diophante effectuait les opérations prescrites dans l'énoncé avant la mise en équation polynomiale du problème. De plus, il envisageait ses polynômes comme des agrégations sans que des opérations y soient incluses.

Περίληψη

Η μεσαιωνική άλγεβρα διακρίνεται από τις άλλες αριθμητικές τεχνικές επιλυσης προβλημάτων τόσο από τη αρχιτεκτονική της λύσης όσο και από το ειδικό λεξιλόγιο που χρησιμοποιεί. Σε μία αλγεβρική επίλυση *ονοματίζονται* ένας ή περισσότεροι άγνωστοι, σαν επακόλουθο των πράξεων επί των αγνώστων το πρόβλημα μεταφέρεται στο τεχνητό περιβάλλον μιας

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εξίσωσης εκπεφρασμένης συναρτήσει των ονοματισμένων δυνάμεων, η οποία στη συνέχεια απλοποιείται και επιλύεται. Στην παρούσα μελέτη εξετάζουμε υπό αυτήν την οπτική τα *Αριθμητικά* του Διοφάντου. Το συμπέρασμα στο οποίο καταλήγουμε είναι ότι η μέθοδος επίλυσης του Διοφάντου ομοιάζει με τη μεσαιωνική άλγεβρα τόσο στο λεξιλόγιο όσο και στην αρχιτεκτονική. Όπως ακριβώς στη μεσαιωνική αραβική και ιταλική άλγεβρα, ο Διόφαντος επεξεργάζεται τις πράξεις που επιτάσσει η εκφώνηση ενός προβλήματος προτού μετατρέψει το πρόβλημα σε πολυωνυμική εξίσωση. Επιπλέον, αντιλαμβανόταν τα πολυώνυμά του ως συναθροισμούς, οι οποίοι δεν περιέχουν καθόλου πράξεις.

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Keywords: Diophantus; Algebra

Next, if from a problem there result certain species equal to similar species . . . it will be necessary to subtract like from like on both sides . . . . If by chance there are on either <side> or on both <sides> wanting species, it will be necessary to add the lacking species on both sides, until the species on both sides become existing, and again to subtract like from like until on each side one species is left.

Diophantus[2]

Diophantus begins his *Arithmetica*[3] with an introduction in which he exposes the technical terms, the notation, the operations with the terms, the steps used to simplify equations, and most significantly the overall structure of what he calls “the way” (*hê hodos*) for solving arithmetical problems. By technical terms, we mean the terms expressing the unknown, called by Diophantus “*arithmos*”, its powers, and its reciprocal powers. By notation, we mean the abbreviations of the terms themselves as well as the sign expressing when a term is “wanting” or “lacking” (called by Diophantus “*leipsis*”).[4] By operations we mean the four arithmetical operations of addition, subtraction, multiplication, and division. By steps we mean those known by their Arabic names *al-jabr* (restoration) and *al-muqābala* (confrontation), by which an equation is simplified so as to receive either the form one term equal to one term, or the form two terms equal to one term. And by structure, we mean the main stages through which the resolution of an arithmetical problem is developed, the distinct parts of “the way”.

The above issues have been studied by one of us in two recent articles, the main aim of which was to demonstrate that there is an intrinsic, organic relation between the above points that Diophantus expounds in the introduction, on the one hand, and the way he solves arithmetical problems in the rest of the book, on the other [Christianidis, 2007; Christianidis, forthcoming]. Meanwhile, the other of us, in a long series of articles, has investigated the conceptual background and the anatomy of the medieval algebraic mode of thought, revealing the subtle, though crucial, differences that the medieval concepts of polynomial and

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[2] This key passage from Diophantus’ introduction to the *Arithmetica* [Tannery, 1893–1895, vol. 1, 14.11–20] summarizes the most essential parts of his method of solution: the transition from problem to equation, and the simplification of the latter. The “next” follows the assigning of names to the unnamed sought-after numbers of the problem and, with the names assigned, working through the operations the problem calls for. One of the objectives of the present paper is to comment upon the content of this excerpt. (*Note on the system of referencing*: the dot (.) separates the page number from the line numbers; thus, the above reference means that the quoted passage is found in page 14 of the volume of the works of Diophantus edited by Tannery in 1893, that is, in the first volume of the *Opera omnia*, and it covers the lines 11–20 of the printed text. Likewise a reference like [Tannery, 1893–1895, vol. 2, 59.15–23] refers to the corresponding page and lines of the second volume of the *Opera omnia*, published in 1895; the former reference is often given simply as [14.11–20].)

[3] There are two editions of the Greek text of the *Arithmetica*, by Tannery [1893–1895], and by Allard [1980], and two editions of the Arabic text, by Sesiano [1982], and by Rashed [1984]. Our analysis is based on the Greek books, and our references are to Tannery’s edition, since Allard’s edition does not offer significant variations to the texts discussed and also because it remains unpublished. Therefore, throughout this paper references to problems from books “IV”, “V”, “VI” refer to the corresponding Greek books, which come after the Books IV to VII in the Arabic translation.

[4] An expression of the form “X *leipsei* Y” (X wanting Y) is usually translated by historians into modern symbols as “$X - Y$”, but as we shall see below, this distorts the meaning of the word. See Section 3.3 and footnotes 24 and 48.