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Research note

The evolution of the concept of homeomorphism

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Abstract

Topology, or analysis situs, has often been regarded as the study of those properties of point sets (in Euclidean space or in abstract spaces) that are invariant under "homeomorphisms." Besides the modern concept of homeomorphism, at least three other concepts were used in this context during the late 19th and early 20th centuries, and regarded (by various mathematicians) as characterizing topology: deformations, diffeomorphisms, and continuous bijections. Poincaré, in particular, characterized analysis situs in terms of deformations in 1892 but in terms of diffeomorphisms in 1895. Eventually Kuratowski showed in 1921 that in the plane there can be a continuous bijection of P onto Q, and of Q onto P, without P and Q being homeomorphic.

Zusammenfassung

Topologie—oder Analysis Situs—wurde oft als Studium solcher Eigenschaften von Punktmengen (im Euklidischen Raum oder in abstrakten Räumen) angesehen, die invariant unter Homöomorphismen sind. Außer dem modernen Konzept des Homöomorphismus wurden während des späten 19. und frühen 20. Jahrhunderts noch mindestens drei Konzepte benutzt und (von mehreren Mathematikern) als charakteristisch für die Topologie angesehen: Deformationen, Diffeomorphismen, und stetige Bijektionen. Poincaré insbesondere charakterisierte Analysis Situs durch Deformationen in 1892 aber dann durch Diffeomorphismen in 1895. Schließlich zeigte Kuratowski in 1921 dass es in der Ebene stetige Bijektionen von P auf Q und von Q auf P geben kann, ohne dass P und Q homöomorph sind.

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1. Introduction

Late in the 20th century, topology (which early in the century, as during the 19th century, had usually been called *analysis situs*) was often presented in textbooks as the study of those properties invariant under "homeomorphisms." The aim of this note is to investigate how the concept of homeomorphism arose and evolved. According to the modern definition, a homeomorphism between two topological spaces X and Y is a one–one function f from X onto Y such that f is continuous and the inverse of f is also continuous. We shall see that, at different times and by different authors, at least four distinct concepts were identified in Euclidean spaces with those mappings under which topological properties were invariant. These were the ideas of deformation, of diffeomorphism, and of one–one continuous

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mapping and the modern idea of homeomorphism. Even in Euclidean spaces, homeomorphisms are more general than deformations (which allow bending and stretching) and diffeomorphisms (where the mappings must be differentiable), while one–one continuous functions are more general than homeomorphisms. All four of these concepts proved useful as topology developed. But it took decades for mathematicians to learn to distinguish clearly between them. In the end, it was the concept of homeomorphism that served in topology in the same way that the concept of isomorphism did in algebra for groups or fields.

Lützen has argued that "in the history of mathematics and science it is often insufficient to consider how concepts are defined; one needs also to consider how they are used. This will often lead to a different and more complex story" [2003, 157]. We have kept this perspective in mind while attempting to unravel the tangled threads of the various concepts related to homeomorphism. We shall see that for certain authors it sheds additional light to look not only at the definition but at how the concept was used, whereas for other authors there remain ambiguities even after this is done.

2. Homeomorphisms and diffeomorphisms

The evolution of the concept of "homeomorphism" was essentially complete by 1935 when Pavel Aleksandrov (Paul Alexandroff) at the University of Moscow and Heinz Hopf at the Eidgenossische Technische Hochschule in Zurich published their justly famous book *Topologie*, aiming to unify the two major branches of topology, the algebraic and the set-theoretic. They took as their fundamental undefined concept "topological space," based on the closure axioms of Kazimierz Kuratowski [1922].[1] And they defined a homeomorphism between topological spaces in the way that is now standard: "A one–one continuous mapping f of a space X into a space Y is called a *topological mapping* or a *homeomorphism* (between X and $f(X) = Y' \subseteq Y$) if the inverse of f is a continuous mapping of Y' to X. Two spaces... are called *homeomorphic* if they can each be mapped topologically onto each other" [Aleksandrov and Hopf, 1935, 52].[2]

Concerning the origins of topology, Aleksandrov and Hopf wrote: "We must regard Poincaré and Cantor as the *immediate* founders of topology" [1935, 5]. So the reader might think that he should read the works of Poincaré and Cantor if he wished to find the origin of the concept of homeomorphism. However, the reader would then find that Cantor's published works contain nothing at all about homeomorphisms, and very little about continuous functions, except for his inadequate proof [1879] of the proposition:

(1) There is no one–one continuous function from a continuous manifold of dimension n to a continuous manifold of dimension m if $m < n$.

But by showing that there is a one–one mapping of n-dimensional Euclidean space onto a line segment, he had established that the number of coordinates does not determine the dimension of the space [1878]. Both E. Netto [1878] and E. Jürgens [1898] believed Cantor's mapping to show that Riemann's 1854 claim—that an n-dimensional manifold is determined by n coordinates—was mistaken. In this context Johnson [1979, 127] writes insightfully about Riemann:

> What we find conspicuously lacking in Riemann's work is the notion of a topological mapping. For modern mathematicians topology is inseparable from homeomorphisms. Riemann never contemplated these in his programme of analysis situs.

Cantor's 1878 article led to numerous attempts to prove (1). Jacob Lüroth [1878] established (1) for $m = 1$ and $m = 2$. Other less successful attempts were made to prove the general case of (1), and eventually a rigorous proof was published by Brouwer [1911].[3] But Cantor's article and those stimulated by it had the effect of making one–one

[1] They showed that Hausdorff's axioms for a topological space were equivalent to Kuratowski's, provided that one omitted Hausdorff's [1914] requirement that any two distinct points are contained in disjoint open sets [Aleksandrov and Hopf, 1935, 43].

[2] Even here it is essential to insist that the homeomorphism is between X and Y', not between X and Y. For if X is the open interval (0, 1) and Y is the closed interval [0, 1], there is a homeomorphism of X with a subset of Y and likewise a homeomorphism of Y with a subset of X, but there is no homeomorphism of X onto Y.

[3] A detailed historical treatment can be found in Dauben [1975] and Johnson [1979, 1981].