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The projective geometry of Mario Pieri: A legacy of Georg Karl Christian von Staudt

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Abstract

The research of Mario Pieri (1860–1913) can be classified into three main areas: metric differential and algebraic geometry and vector analysis; foundations of geometry and arithmetic; logic and the philosophy of science. In writing this article, I intend to reveal some important aspects of his contributions to the foundations of projective geometry, notably those that emanated from his intensive study of the works of Georg Karl Christian von Staudt (1798–1867). Pieri was the first geometer to successfully establish projective geometry as an independent subject (rigorous mathematical theory), freed from all ties to Euclidean geometry. The path to this achievement began with Staudt, and involved the reformulation of the classical ideas of cross ratio and projectivity in terms of harmonic sets, as well as a critical analysis of the proof of a fundamental theorem that connects these ideas. Included is a brief overview of Pieri's life and work.

Sommario

La ricerca di Mario Pieri (1860–1913) può essere classificata in tre zone principali: differenziale metrico e la geometria algebrica ed analisi di vettore; fondamenti della geometria e dell'aritmetica logica e la filosofia di scienza. Nel questo articolo, intendo rivelare alcune funzioni importanti dei suoi contributi ai fondamenti della geometria proiettiva, considerevolmente quelli che sono derivato dal suo studio intenso sugli impianti di Georg Karl Christian von Staudt (1798–1867). Pieri era il primo geometra per stabilire con successo la geometria proiettiva come oggetto indipendente, liberato da tutti i legami alla geometria euclidea. Il percorso a questo successo ha cominciato con Staudt ed ha coinvolto la nuova formulazione delle ideee classiche del rapporto e del projectivity trasversali in termini di insiemi armonici, così come un'analisi critica della prova di un teorema fondamentale che collega queste idee. Inclusa è una breve descrizione di vita e de lavoro del Pieri.

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1. Introduction

The research of Mario Pieri (1860–1913) can be classified into three main areas: (1) metric differential and algebraic geometry and vector analysis; (2) foundations of geometry and arithmetic; (3) logic and the philosophy of science. In writing this article, I intend to reveal some important aspects of his contributions to the foundations of projective geometry, notably those that emanated from his intensive study of the works of Georg Karl Christian von Staudt (1798–1867).

Pieri was the first geometer to successfully establish projective geometry as an independent subject, freed from all ties to Euclidean geometry. The path to this achievement began with Staudt, whose famous *Geometrie der Lage* [1847] was intended to rid projective geometry of metric concepts. In this endeavor, the idea of a harmonic set played a pivotal role. Staudt appealed to it in reformulating two concepts of classical projective geometry: the cross ratio, which until his time had been defined metrically; and the projectivity, a transformation that had classically been defined in terms of sequences of correspondences called perspectivities. A fundamental theorem on projectivities (which ensures that the transformation is completely determined by assigning distinct collinear images to three distinct collinear points) intercedes in both of these ideas. This theorem is important not only to reconcile Staudt's reformulated definitions of cross ratio and projectivity with their classical definitions, but also because it simplifies the proofs of other projective theorems, enabling them to be constructed without appealing to concepts extraneous to the theorems themselves (see, for example, [Coxeter, 1964, 34–35]). Staudt's proof of the fundamental theorem, however, was one of the reasons that he did not achieve his goal of a metric-free projective geometry. Pieri would bring Staudt's dream to fruition 50 years later with his axiomatization of projective geometry [Pieri, 1897–1898].

Before I trace the evolution of ideas beginning with [von Staudt, 1847] and culminating in [Pieri, 1897–1898] and its modifications, I give a brief overview in Section 2 of Pieri's life and work.[1] In Section 3, I discuss the classical concept of projection and section and relate this process to the construction of projectivities as finite sequences of perspectivities. In Section 4, I set the context for Staudt's attempts to achieve a metric-free definition of projective geometry, by showing how he reformulated the classical definition of cross ratio into a purely projective one. To do this, I introduce the quadrangle, a fundamental plane figure of projective geometry, and demonstrate how Staudt used it to construct harmonic sets. In Section 5, I show the role of harmonic sets in Staudt's definition of projectivity. The fundamental theorem enters into the discussion in two ways: to ensure that Staudt's cross ratio is well defined, and to equate Staudt's definition of projectivity in terms of harmonic sets with the classical one in terms of

[1] In-depth studies of Pieri's life and research are currently being written. The following three books will be published by Birkhäuser, Boston: *The Legacy of Mario Pieri: Geometry and Arithmetic* [Marchisotto and Smith], *The Legacy of Mario Pieri: Logic and Geometry* [Marchisotto et al.], and *The Legacy of Mario Pieri: Differential and Algebraic Geometry* [Marchisotto and Smith].