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George Boole and the origins of invariant theory

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Abstract

Historians have repeatedly asserted that invariant theory was born in two papers of George Boole (1841 and 1842). Although several themes and techniques of 19th-century invariant theory are enunciated in this work, in reacting to it (and thereby founding the British school of invariant theory), Arthur Cayley shifted Boole's research program.

Résumé

Les historiens ont dit à plusiers reprises que la théorie des invariants est née dans deux communications de George Boole (1841 et 1842). Bien que plusiers thèmes de la théorie des invariants du dix-neuvième siècle sont énoncés dans cet ouvrage, en répondant (et fondant l'école britannique de la théorie des invariants) Arthur Cayley a donné un changement au programme de recherche de Boole.

MSC: 01A55; 13A50; 15A72; 15-03; 12-03

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1. Introduction

One of the earliest British researchers on invariant theory remarked that

> What I have called Modern Algebra [i.e., invariant theory] may be said to have taken its origin from a paper in the Cambridge Mathematical Journal for Nov. 1841, where Dr. Boole established the principles [of invariance] just stated and made some important applications of them. [Salmon, 1885, 109]

Soon thereafter,W.-F. Meyer offered a similar opinion in his report on the current state of invariant theory to the Deutschen Mathematiker-Vereinigung [Meyer, 1892, 82], and in his preface to the French translation of this work, Maurice d'Ocagne remarked:

> Aside from some isolated remarks... no result of importance was obtained before Boole who, in 1841, introduced a whole series of expressions of this type [i.e., invariants]. But credit undeniably goes to Cayley of having first begun the systematic calculation of invariants, to which, from 1846 to 1871, he devoted a sequence of important Memoirs whose entirety forms a monument. [Meyer, 1897, 4]

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These 19th-century judgements are confirmed by modern experts (see Parshall, 1989, 185, and Crilly, 1986, 241–243 and 1994, 788). Cayley's reaction to Boole's paper reveals a shift of emphasis, however, showing the growing maturity of an independent theory of invariants.

2. Background to Boole's papers

George Boole had learned higher mathematics by independent study, especially of Newton's *Principia* and Lagrange's *Méchanique analitique*. His paper of 1841 and its continuation in 1842 explicitly generalize a calculation from the *Méchanique analitique*. Boole began his 1841 paper by observing that, "The transformation of homogeneous functions by linear substitutions, is an important and oft-recurring problem of analysis. In the *Méchanique Analytique* of Lagrange, it occupies a very prominent place..." [Boole, 1841b, 1].

We may find this transformation where Lagrange discussed the rotation of various kinds of bodies. After briefly discussing freely moving bodies, Lagrange turned to a discussion of suspended bodies. As before, he considered the kinetic energy T, first showing how to describe the motion of the body if T is a sum of squares. He then showed—what was of interest to Boole in his paper—that the general case "can be reduced to the preceding one by means of some substitutions" [Lagrange, 1788, 395]. That is, by a linear substitution

$$\begin{aligned} p &= p'x + p''y + p'''z, \\ q &= q'x + q''y + q'''z, \\ r &= r'x + r''y + r'''z \end{aligned}$$

one can express

$$T = \frac{1}{2}\left(Ap^2 + Bq^2 + Cr^2\right) - Fqr - Gpr - Hpq \tag{1}$$

as

$$T = \frac{1}{2}\left(\alpha x^2 + \beta y^2 + \gamma z^2\right). \tag{2}$$

See Parshall [1989, 159–160]. Lagrange determined the coefficients of the linear substitution and also the coefficients α, β, and γ of the diagonalized form by solving a cubic equation whose roots are α, β, and γ [Lagrange, 1788, 397].

As Boole further noted in the introductory paragraph of his paper, others besides Lagrange had worked on the algebraic problem of reducing a quadratic form to a sum of squares—Hawkins [1975] gives us a detailed account of these developments to Cauchy and beyond—and the most general conclusion so far reached was

> that it is always possible to take away the products of the variables $x_1, x_2, \ldots, x_m$, from a proposed homogenous function of the second degree, Q, by the linear substitution of a new set of variables, $y_1, y_2, \ldots, y_m$, connected with the original ones by the relation
>
> $$x_1^2 + x_2^2 + \cdots + x_m^2 = y_1^2 + y_2^2 + \cdots + y_m^2 \ldots (1);$$
>
> or in other words, to determine, subject to (1), the values of the coefficients $A_1, A_2, \ldots, A_m$, in the equation of transformation,
>
> $$Q = A_1 y_1^2 + A_2 y_2^2 + \cdots + A_m y_m^2 \ldots (2).$$
>
> [Boole, 1841b, 1–2]

In modern language, it is always possible to diagonalize a quadratic form *via* an orthogonal transformation.

Boole then proposed his own method of analysis of the problem as an alternative to the prevailing method.