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Note

New upper bound for multicolor Ramsey number of odd cycles[☆]

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ABSTRACT

Let $r_k(C_{2m+1})$ be the k-color Ramsey number of an odd cycle C_{2m+1} of length $2m+1$. It is shown that for each fixed $m \ge 2$,

$$r_k(C_{2m+1}) < c^k \sqrt{k!}$$

for all sufficiently large k, where $c = c(m) > 0$ is a constant. This improves an old result by Bondy and Erdős (1973).

1. Introduction

Let G be a graph. The multicolor *Ramsey number* $r_k(G)$ is defined as the minimum integer N such that each edge coloring of the complete graph K_N with k colors contains a monochromatic G as a subgraph. The Turán number $ex(N; G)$ is the maximum number of edges among all graphs of order N that contain no G. For the complete bipartite graph $K_{t,s}$ with $s \ge t$, a well known argument of Kővári, Sós, and Turán [15] gives that $ex(N; K_{t,s}) \le \frac{1}{2}\left[(s-1)^{1/t}N^{2-1/t} + (t-1)N\right]$. For large N, the upper bound was improved by Füredi [11] to $\frac{1}{2}((s-t+1)^{1/t}+o(1))N^{2-1/t}$. Let $N = r_k(K_{t,s})-1$. Since there exists a k-coloring of the edges of K_N such that it contains no monochromatic $K_{t,s}$, each color class can have at most $ex(N; K_{t,s})$ edges. Thus $\binom{N}{2} \le k \cdot ex(N; K_{t,s})$. From an easy calculation, we have $r_k(K_{t,s}) \le (s-t+1+o(1))k^t$ as $k \to \infty$. Hence $r_k(G)$ can be bounded from above by a polynomial of k if G is a bipartite graph.

However, the situation becomes dramatically different when G is non-bipartite. Denote $r_k(K_3)$ by $r_k(3)$ for short. An old problem proposed by Erdős is to determine

$$\lim_{k\to\infty} (r_k(3))^{1/k}.$$

It is known from Chung [3] that $r_k(3)$ is super-multiplicative in k so that $\lim_{k\to\infty}(r_k(3))^{1/k}$ exists. Up to now, we only know that

$$1073^{k/6} \le r_k(3) \le c \cdot k!,$$

where $c > 0$ is a constant, see [1,4,8,10,20] and their references for more details.

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Let C_{2m+1} be an odd cycle of length $2m+1$. For $m=1$, the multicolor Ramsey number $r_k(3)$ has attracted a lot of attention. For general fixed integer $m \geq 2$, Erdős and Graham [7] showed that

$$m2^k < r_k(C_{2m+1}) < 2(k+2)!m. \tag{1}$$

Bondy and Erdős [2] observed that

$$m2^k + 1 \leq r_k(C_{2m+1}) \leq (2m+1)\cdot(k+2)!. \tag{2}$$

For the lower bound, a recent result by Day and Johnson [6] gives that for $m \geq 2$, there exists a constant $\epsilon = \epsilon(m) > 0$ such that $r_k(C_{2m+1}) > 2m\cdot(2+\epsilon)^{k-1}$ for all large k. The upper bound was improved by Graham, Rothschild and Spencer [12] to $r_k(C_{2m+1}) < 2m\cdot(k+2)!$. In particular, for $m=2$, Li [16] showed that $r_k(C_5) \leq c\sqrt{18^k k!}$ for all $k \geq 3$, where $0 < c < 1/10$ is a constant. However, there has not been substantial progress on the value of $r_k(C_{2m+1})$ for $m \geq 3$.

Let us point out that the situation is much different when k is fixed. For $k=2$, Bondy and Erdős [2], Faudree and Schelp [9] and Rosta [19] independently obtained that $r_2(C_{2m+1}) = 4m+1$ for all $m \geq 2$. For $k=3$, Łuczak [18] proved that $r_3(C_{2m+1}) = (8+o(1))m$ as $m \to \infty$ by using the regularity lemma. Kohayakawa, Simonovits and Skokan [14] used Łuczak's method together with stability methods to prove that $r_3(C_{2m+1}) = 8m+1$ for sufficiently large m. Recently, Jenssen and Skokan [13] established that $r_k(C_{2m+1}) = 2^k m + 1$ for all fixed k and sufficiently large m.

In this short note, we have an upper bound for $r_k(C_{2m+1})$ as follows.

Theorem 1. *Let $m \geq 2$ be a fixed integer. We have*

$$r_k(C_{2m+1}) < c^k\sqrt{k!}$$

for all sufficiently large k, where $c = c(m) > 0$ is a constant.

Remark. We do not attempt to optimize the constant $c = c(m)$ in the above theorem, since we care more about the exponent of $k!$.

Let $N = r_k(G) - 1$. From the definition, there exists a k-edge coloring of K_N containing no monochromatic G. In such an edge coloring, any graph induced by a monochromatic set of edges is called a Ramsey graph. Let $\epsilon > 0$ be a constant. Under the assumption that each Ramsey graph H for $r_k(C_{2m+1})$ has minimum degree at least $\epsilon d(H)$ for large k, Li [16] showed that $r_k(C_{2m+1}) \leq \left(c^k k!\right)^{1/m}$, where $d(H)$ is the average degree of H and $c = c(\epsilon, m) > 0$ is a constant.

2. Proof of the main result

In order to prove Theorem 1, we need the following well-known result.

Theorem 2 (*Chvátal [5]*)**.** *Let T_m be a tree of order m. We have*

$$r(T_m, K_n) = (m-1)(n-1)+1.$$

For a graph G, let $\alpha(G)$ denote the independence number of G.

Lemma 1 (*Li and Zang [17]*)**.** *Let $m \geq 2$ be an integer and let $G = (V, E)$ be a graph of order N that contains no C_{2m+1}. We have*

$$\alpha(G) \geq \frac{1}{(2m-1)2^{(m-1)/m}}\left(\sum_{v\in V} d(v)^{1/(m-1)}\right)^{(m-1)/m},$$

where $d(v)$ is the degree of v in graph G.

Proof of Theorem 1. Let $m \geq 2$ and $k \geq 3$ be integers. For convenience, let $r_k = r_k(C_{2m+1})$ and $N = r_k - 1$. Let $K_N = (V, E)$ be the complete graph on vertex set V of order N. From the definition, there exists an edge-coloring of K_N using k colors such that it contains no monochromatic C_{2m+1}. Let E_i denote the monochromatic set of edges in color i for $i = 1, 2, \ldots, k$. Without loss of generality, we may assume that E_1 has the largest cardinality among all E_i's. Therefore $|E_1| \geq \binom{N}{2}\big/k$. Let G be the graph with vertex set V and edge set E_1. Then the average degree d of G satisfies

$$d = \frac{2|E_1|}{N} \geq \frac{N-1}{k} = \frac{r_k - 2}{k}.$$

Consider an independent set I of G with $|I| = \alpha(G)$. Since any edge of K_N between two vertices in I is colored by one of the colors $2, 3, \ldots, k$, the subgraph induced by I is an edge-colored complete graph using $k-1$ colors, which contains no monochromatic C_{2m+1}. Thus $|I| \leq r_{k-1} - 1$, and thus Lemma 1 implies that

$$r_{k-1} - 1 \geq a\left(\sum_{v\in V} d(v)^{1/(m-1)}\right)^{(m-1)/m}, \tag{3}$$

where $a = a(m)$ is a constant.