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Improved bounds for rainbow numbers of matchings in plane triangulations

Zhongmei Qin [a], Yongxin Lan [b,*], Yongtang Shi [b]

[a] School of Science, Chang'an University, Xi'an, Shaanxi 710064, PR China
[b] Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, PR China

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ABSTRACT

Given two graphs G and H, the *rainbow number* $rb(G, H)$ for H with respect to G is defined as the minimum number k such that any k-edge-coloring of G contains a rainbow H, i.e., a copy of H, all of whose edges have different colors. Denote by kK_2 a matching of size k and $\mathcal{T}_n$ the class of all plane triangulations of order n, respectively. In Jendrol' et al. (2014), the authors determined the exact values of $rb(\mathcal{T}_n, kK_2)$ for $2 \le k \le 4$ and proved that $2n+2k-9 \le rb(\mathcal{T}_n, kK_2) \le 2n+2k-7+2\binom{2k-2}{3}$ for $k \ge 5$. In this paper, we improve the upper bounds and prove that $rb(\mathcal{T}_n, kK_2) \le 2n + 6k - 16$ for $n \ge 2k$ and $k \ge 5$. Especially, we show that $rb(\mathcal{T}_n, 5K_2) = 2n + 1$ for $n \ge 11$.

1. Introduction

All graphs in this paper are undirected, finite and simple. We follow [3] for graph theoretical notation and terminology not defined here. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two disjoint subsets X and Y of $V(G)$, we use $E_G(X, Y)$ to denote the set of edges of G that have one end in X and the other in Y. We also denote $E_G(X, X) = E_G(X)$. Let $e(G)$ denote the number of edges of G, $e_G(X, Y)$ the number of edges of $E_G(X, Y)$, $e_G(X)$ the number of edges of $E_G(X)$. If $X = \{x\}$, then we write $E_G(x, Y)$ and $e_G(x, Y)$, respectively. For a vertex $x \in V(G)$, we use $N_G(x)$ to denote the set of vertices in G which are adjacent to x. We define $d_G(x) = |N_G(x)|$. Given vertex sets $X, Y \subseteq V(G)$, the subgraph of G induced by X, denoted $G[X]$, is the graph with vertex set X and edge set $\{xy \in E(G) : x, y \in X\}$. We denote by $Y\backslash X$ the set $Y - X$.

A subgraph of an edge-colored graph is *rainbow* if all of its edges are colored distinct. Given two graphs G and H, the *rainbow number* $rb(G, H)$ for H with respect to G is defined as the minimum number k such that any k-edge-coloring of G contains a rainbow copy of H. When $G = K_n$, the rainbow number is closely related to anti-Ramsey number, which was introduced by Erdős, Simonovits and Sós [5] in 1975. The *anti-Ramsey number*, denoted by $f(K_n, H)$, is the maximum number c for which there is a way to color the edges of K_n with c colors such that every subgraph H of K_n has at least two edges of the same color. Clearly, $rb(K_n, H) = f(K_n, H) + 1$.

Let $\mathcal{T}_n$ denote the class of all plane triangulations of order n. We denote by $rb(\mathcal{T}_n, H)$ the minimum number of colors k such that, if $H \subseteq T_n \in \mathcal{T}_n$, then any edge-coloring of T_n with at least k colors contains a rainbow copy of H. The rainbow number has been widely studied. The rainbow numbers for matchings with respect to complete graph have been completely determined step by step in [4–6,22]. Also, the rainbow numbers for some other special graph classes in complete graphs have been obtained, see [1,8,11–14,20]. Meanwhile, the researchers studied the rainbow number when host graph changed from the complete graph to others, such as complete bipartite graphs [2,18], planar graphs [9,10,15,17], hypergraphs [21], etc. For more results on rainbow numbers, we refer to the survey [7].

* Corresponding author.
E-mail addresses: qinzhongmei90@163.com (Z. Qin), yxlan0@126.com (Y. Lan), shi@nankai.edu.cn (Y. Shi).

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In this paper we study the rainbow number when host graphs are plane triangulations. Let $\mathcal{T}_n$ be the family of all plane triangulations on n vertices. As one of the most important structures in graphs, the study of rainbow number in plane triangulations $rb(\mathcal{T}_n, H)$ was initiated by Horňák et al. [9]. Horňák et al. [9] investigated the rainbow numbers for cycles. Very recently, Lan, Shi and Song [17] improve some bounds for the rainbow number of cycles, and also get some results for paths. Jendrol′, Schiermeyer and Tu [10] investigated the rainbow numbers for matchings in plane triangulations. We summarize their results as follows, where kK_2 denotes a matching of size k.

Theorem 1.1 ([10]).

(1) $2n + 2k - 9 \le rb(\mathcal{T}_n, kK_2) \le 2n + 2k - 7 + 2\binom{2k-2}{3}$ *for all* $k \ge 5$.

(2) $rb(\mathcal{T}_n, 2K_2) = \begin{cases} 4 & n = 4; \\ 2 & n \ge 5. \end{cases}$

(3) $rb(\mathcal{T}_n, 3K_2) = \begin{cases} 8 & n = 6; \\ n+1 & n \ge 7. \end{cases}$

(4) $rb(\mathcal{T}_n, 4K_2) = 2n - 1$ *for all* $n \ge 8$.

Recently, Jin and Ye [15] investigated the rainbow numbers of kK_2 in the maximal outerplanar graphs. In this paper, we improve the upper bounds and prove that $rb(\mathcal{T}_n, kK_2) \le 2n + 6k - 16$ for $n \ge 2k$ and $k \ge 5$. Especially, we show that $rb(\mathcal{T}_n, 5K_2) = 2n + 1$ for $n \ge 11$ by using the method of Jendrol′, Schiermeyer and Tu [10].

Theorem 1.2. *For* $n \ge 2k$ *and* $k \ge 5$, $rb(\mathcal{T}_n, kK_2) \le 2n + 6k - 16$.

Theorem 1.3. *For* $n \ge 11$, $rb(\mathcal{T}_n, 5K_2) = 2n + 1$.

The following theorem will be used in our proof. A graph G is called *factor-critical* if $G - v$ contains a perfect matching for each $v \in V(G)$. A graph is called *hypoHamiltonian* if for every vertex u, $G - u$ is Hamiltonian.

Theorem 1.4 ([19]). *Given a graph* $G = (V, E)$ *and* $|V| = n$, *let* d *be the size of a maximum matching of* G. *Then there exists a subset* S *with* $|S| \le d$ *such that*

$$d = \frac{1}{2}(n - (o(G - S) - |S|)),$$

where $o(H)$ *is the number of components in the graph* H *with an odd number of vertices. Moreover, each odd component of* $G - S$ *is factor-critical.*

Lemma 1.5. *Let* G *be a planar triangulation on* $n \ge 4$ *vertices. Then*

(a) [10,16] for $5 \le n \le 7$, *G is hypoHamiltonian.*
(b) G is 3-connected.

2. Proof of Theorem 1.2

By induction on k. The statement is true for $k \le 4$ by Theorem 1.1. Now we assume $k \ge 5$. Let T_n be a plane triangulation on n vertices. By contradiction, let c be an edge-coloring of T_n with $2n + 6k - 16$ colors such that T_n does not contain any rainbow kK_2. Let G be a rainbow spanning subgraph of T_n with $2n + 6k - 16$ edges. Then G is kK_2-free. Since $2n + 6k - 16 > 2n + 6(k - 1) - 16$, G contains a $(k - 1)K_2$ by the induction hypothesis. Let $u_1w_1, u_2w_2, \ldots, u_{k-1}w_{k-1}$ be a $(k-1)K_2$ of G, and let H be an induced subgraph by $\{u_1, \ldots, u_{k-1}, w_1, \ldots, w_{k-1}\}$ in G. Then $e_G(H) \le 3(2k-2)-6 = 6k-12$.

Let $R = V(G)\backslash V(H)$. Since G is kK_2-free, $E(G[R]) = \emptyset$. Then we have $G - E_G(H)$ is a bipartite planar graph with n vertices, which implies $e_G(V(H), R) \le 2n - 4$. Thus, $e(G) = e_G(H) + e_G(V(H), R) \le 6k - 12 + 2n - 4 = 2n + 6k - 16$. Since $e(G) = 2n + 6k - 16$, we have $e_G(H) = 6k - 12$ and $e_G(V(H), R) = 2n - 4$. Hence, $G[V(H)]$ is a plane triangulation with $2k - 2$ vertices and $G - E_G(H)$ is a maximal bipartite planar graph with n vertices. Since $u_1w_1 \in E(G)$, there must exist a quadrangular face with vertices u_1, r_1, w_1, r_2 in order in $G-E_G(H)$, where $r_1, r_2 \in R$. But then the graph induced by the edges u_1r_1, w_1r_2, u_2w_2,…, $u_{k-1}w_{k-1}$ is a rainbow subgraph of T_n isomorphic to kK_2, a contradiction.

3. Proof of Theorem 1.3

By Theorem 1.1, we only need to show that $rb(\mathcal{T}_n, 5K_2) \le 2n+1$. Suppose $rb(\mathcal{T}_n, 5K_2) \ge 2n+2$. Then there exists a plane triangulation T_n on n vertices containing no rainbow $5K_2$ under an edge-coloring c used $2n+1$ colors. Let $G \subset T_n$ be a rainbow spanning subgraph with $2n + 1$ edges. Then G has no a copy of $5K_2$. By Theorem 1.1, G has a copy of $4K_2$. By Theorem 1.4, there exists an $S \subseteq V(G)$ with $|S| = s \le 4$, such that $q = o(G - S) = n - 8 + s$. Let $A_1, \ldots, A_q$ be all the odd components of $G - S$. Assume $|V(A_i)| = a_i$ for each $i \in [q]$ and $a_1 \ge a_2 \ge \cdots \ge a_q$. Let $t = \min\{i : a_i = 1\}$ and $V_0 = \{v_t, \ldots, v_q\}$, where $v_j \in V(A_j)$. Assume $d_G(v_t) \ge d_G(v_{t+1}) \ge \cdots \ge d_G(v_q)$. Let B denote the set of vertices of all the even components of $G - S$. We first prove a useful claim.