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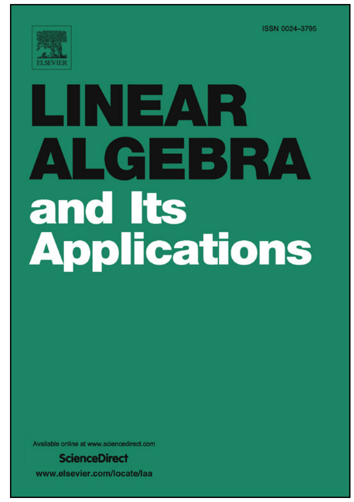
Ravindra B. Bapat, Hiroshi Kurata



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On Cartesian product of Euclidean distance matrices [*]

Ravindra B. Bapat [†] Hiroshi Kurata [‡]

Abstract

If $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$, we define the product $A \oslash B$ as $A \oslash B = A \otimes J_n + J_m \otimes B$, where $\otimes$ denotes the Kronecker product and J_n is the $n \times n$ matrix of all ones. We refer to this product as the Cartesian product of A and B since if D_1 and D_2 are the distance matrices of graphs G_1 and G_2 respectively, then $D_1 \oslash D_2$ is the distance matrix of the Cartesian product $G_1 \square G_2$. We study Cartesian products of Euclidean distance matrices (EDMs). We prove that if A and B are EDMs, then so is the product $A \oslash B$. We show that if A is an EDM and U is symmetric, then $A \otimes U$ is an EDM if and only if $U = cJ_n$ for some c. It is shown that for EDMs A and B, $A \oslash B$ is spherical if and only if both A and B are spherical. If A and B are EDMs, then we derive expressions for the rank and the Moore-Penrose inverse of $A \oslash B$. In the final section we consider the product $A \oslash B$ for arbitrary matrices. For $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times n}$, we show that all nonzero minors of $A \oslash B$ of order $m+n-1$ are equal. An explicit formula for a nonzero minor of order $m+n-1$ is proved. The result is shown to generalize the familiar fact that the determinant of the distance matrix of a tree on n vertices does not depend on the tree and is a function only of n.

AMS Classification: 15B48, 05C05

1 Introduction

Let $\mathbb{R}^{n\times n}$ denote the set of real $n \times n$ matrices. If $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$, we define the product $A \oslash B$ as

$$A \oslash B = A \otimes J_n + J_m \otimes B,$$

where $\otimes$ denotes the Kronecker product and J_n is the $n \times n$ matrix of all ones. We refer to this product as the Cartesian product of A and B, since it is related to Cartesian product of graphs as we will see shortly.

The Cartesian product is associative, that is, for $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times p}$, it holds that

$$(A \oslash B) \oslash C = A \oslash (B \oslash C).$$

[*]Key Words: Euclidean distance matrix, Predistance matrix, Positive semidefinite matrix, Cartesian product of graphs, Moore-Penrose inverse, Tree.

[†]Indian Statistical Institute, New Delhi, 110016, India. (rbb@isid.ac.in)

[‡]Corresponding author. Graduate School of Arts and Sciences, The University of Tokyo, Tokyo, 153-8902, Japan. (kurata@waka.c.u-tokyo.ac.jp)