Studies in History and Philosophy of Science 44 (2013) 20–31

Contents lists available at SciVerse ScienceDirect

Studies in History and Philosophy of Science

journal homepage: www.elsevier.com/locate/shpsa

Extension and measurement: A constructivist program from Leibniz to Grassmann

Erik C. Banks

Department of Philosophy, Wright State University, 3640 Colonel Glenn Hwy, Dayton, OH 45435, USA

ARTICLE INFO

Article history:
Received 16 February 2012
Received in revised form 22 June 2012
Available online 26 November 2012

Keywords:
Leibniz
Grassmann
Riemann
Extension
Algebra
Space

ABSTRACT

This paper traces what I see as a Leibniz-inspired constructivist program through the eyes of the 19th century philosopher-mathematicians Herbart, Riemann and Grassmann, and then uses Grassmann's algebra of points to build up levels of extension algebraically. The connection between extension and measurement is investigated in line with this constructivist program.

When citing this paper, please use the full journal title *Studies in History and Philosophy of Science*

1. Introduction

Extension is probably the most general natural property known. But is it a simple, fundamental property or is it constructed out of still *more* fundamental properties or relations? Descartes claimed, along with most philosophers and scientists, that extension *was* a simple property that could neither be explained nor constructed. Leibniz denied that extension was a fundamental property, and followers of Leibniz, such as Herbart, sought an explicit construction of extension from unextended points and forces in a physical monadology. Finally the mathematicians Riemann and Grassmann took a turn at developing extension from scratch, without assuming the property as fundamental. I will trace this constructivist line of argument to Grassmann's algebra of extension and show how to interpret it in this light. Constructivism is a minority view, but its story is worth telling, for, despite the magnitude of the names involved in some way with this tradition (Leibniz, Herbart, Riemann and Grassmann), the problem of constructing extension has attracted very little interest among philosophers or historians and is long overdue for a taking stock.

2. Extension: Concept or Intuition?

The first problem is to define extension. What definitions there are, from the medievals to Kant, talk about the *apartness* of parts of magnitudes like a length or a time, the parts of an object, stages of a process and so forth. The opposite is an intensive magnitude with a degree or intensity, but no extended parts. These definitions are clearly circular, since 'apartness' assumes the intuition we are trying to define. Also, a length and a degree, say of temperature or density, are both representable in the same way by the real numbers $\mathcal{R}$. At no point is any appeal made to the outside-ness or inside-ness of parts or degrees. The intuition of extension plays *no role* in the mathematical construction of the reals; it rather has to do with how we *represent* the real continuum. Second, the distinction between discrete and continuous magnitude plays no role here either, since extensions can be either discrete or continuous.

Generally speaking, mathematics makes no distinction whatever between an extended representation, say of the real numbers, and any other interpretation. You could think of sets as points collected in extended space, or as anything else you wish. Nowhere in

E-mail address: erik.banks@wright.edu

0039-3681/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.shpsa.2012.07.011

mathematics is it ever specified what *would* make for an extended interpretation. When a specific spacetime structure is built up, we assume a blob-like extension of points, call it $\mathcal{R} \times \mathcal{R} \times \mathcal{R} = \mathcal{R}^3$. Then we define a topology for the blob (an open ball, a sphere with boundary, a torus) by laying constraints on neighborhoods around points. A coordinate mesh can be imposed, the curvature and the metric can be defined, all the way up to the causal structure of lightcones in the space, but nowhere has the assumption of the extended blob been justified as opposed to any other interpretation of the sets making up $\mathcal{R}^3$. We are forced to conclude that extension, just like sense or order,[1] is not defined in mathematics, rather it is an intuition we bring to mathematics when we make an extended representation of its abstract structure.

This brings us to Kant's view, that the extended 'drafting board' of intuition, on which all extended constructions are done, is necessarily *assumed* and cannot be analyzed, or brought under concepts, because any conceptual analysis would assume extension and beg the question. Assuming one tried to generate a dynamical construction of space from causal interactions of spaceless, timeless entities like monads, it seems one would be bound to assume a causal interaction over space and time *between* the monads, so that the construction must fail. It has even been proposed that *all* dynamical, causal derivations of space and time are doomed to failure, even in advanced physics, since dynamics and causation simply assume extended length. If this were true, Kant would be right to call extension an intuition we bring to all extended representations of the world, rather than a concept, for apparently we have no conceptual grasp of it at all.[2]

3. Evidence for a Leibniz-Inspired Constructivist Program

A philosophico-scientific program for a construction of extension certainly existed in Germany, at least up until the time of J.F. Herbart, Riemann and Grassmann. The construction originally took the form of what was called a physical monadology, inspired by Leibniz's writings, and pursued by Christian Wolff, Roger Boscovich and the Pre-Critical Kant among others (Kant 2003; Friedman 1998). As we shall see in more detail below, Herbart also developed such a physical monadology in his *Allgemeine Metaphysik* of 1828-29, which involved unextended point-like *Wesen* (beings, entities) and instantaneous forces, and where extension is traced out by an associative-dissociative diffusion of forces. Herbart, in turn, is a direct historical link to the German mathematician Riemann (Banks 2005; Scholz 1982) and is even mentioned by name alongside Gauss in Riemann's seminal 1854 *Probevorlesung* on the foundations of geometry. Grassmann looked back to Leibniz and claimed that that thegeometric algebra developed in his groundbreaking 1844 *Ausdehnungslehre* was the natural development of Leibniz's ideas on the "geometric characteristic," announced in the letter to Huygens in 1679. Grassmann's prize essay of 1847 was written in response to a challenge to complete Leibniz's project for the characteristic.[3] Grassmann claimed that because his geometric algebra described abstract entities and relations at a level *prior* to extension, they could be used to *analyze* the concept of extension, where sensory human visualizations could no longer reach. Grassmann, writing about himself in the third person, also attributed this goal to Leibniz:

> Finally, at the end of Leibniz's presentation [the 1679 letter] is yet another remarkable point where he quite clearly expresses the applicability of this analysis to objects that are not of a spatial nature, but adds that it is not possible to give a clear concept of this in a few words. Now, in fact, as is demonstrated throughout Grassmann's *Ausdehnungslehre*, all concepts and laws of the new analysis can be developed completely independently of spatial intuitions, since they can be tied to the abstract concept of a continuous transformation; and, once one has grasped this idea of a pure, conceptually interpreted continuous transformation, it is easy to see that the laws developed in this essay are also capable of this interpretation, stripped of spatial intuitions. (Grassmann 1995, p. 384)

4. Herbart: A Transitional Figure

J.F. Herbart (1776-1841) was Professor at Königsberg and Göttingen. He was famous for his mathematical psychology of psychic forces and for his philosophy of spatial representation. Herbart was a realistic, scientifically minded philosopher who advocated constructivist methods and who challenged the Kantian a priori drafting-board intuition of space and time extension, believing he could analyze it further. Herbart is often considered a Leibnizian for this reason, because he worked on constructing space in a physical monadology, like Christian Wolff and the young Kant before him. In his *Metaphysik*, Herbart describes point-like Wesen possessed of instantaneous forces. He then describes associative-dissociative extension-tracing processes among the Wesen that amount to a construction of extension. Herbart also believed, as Leibniz had, that the primary forces of nature, at this metaphysical level, were instantaneous and did not require an extended representation. What could he have meant by this? Of course if we simply take the notion of force straight from mechanics it is not true: a freely acting force always requires some space to act through (producing energy) and some time to act over (producing momentum) and manifests in accelerations. However, as the natural philosopher and metaphysician Roger Boscovich pointed out (1966), we can imagine that there *are* no free forces at the primary level, but instead that each primary 'active' force always acts against a countervailing 'passive' force. Boscovich proposes that the greatest active force is equilibrated by the smallest passive force in a small enough 'snapshot' of time and space. Actually there are two possibilities: 1) either the stronger force will always overcome the weaker continuously and there never is a minimum, or else 2) there are discrete extended quanta of energy-momentum which *cannot* be made smaller, even when the greatest force acts against the weakest. Beneath that extended level, all is in stasis: we can only assemble extended quanta out of static, combinatorial patterns of primary forces in equilibrium.

Let us suppose this balance point to have been reached. Herbart now sees two kinds of instantaneous relationships for the primary forces: they either depend on each another (Zusammen), or they do not (Nicht-Zusammen). When the forces are put into a static combinatorial pattern, their instantaneous relationships, which we express in symbols, can bear the intuitive interpretation of the serial tracing of an extension, see Figure 1 below.

[1] Directions, such as right and left, have to be given by some means external to mathematics, if we are to avoid begging the question. Order can be represented, say by odd and even permutations, if we *already* understand the order 123 in the positive sense, i.e. the handedness of a coordinate system, where one stands in the origin and counts the axes off in clockwise (right handed) or counterclockwise (left handed) fashion. Similarly, the ordered pair <u,v> can be defined as being in the *same* order as <x,y>=<u,v> if one *already* understands what the order of <x,y> is. Russell (1903) proposes including order or sense among the undefined notions, but says nothing about extension.

[2] Kant does however think that buried somehow in extended intuition is actual knowledge that can be teased out indirectly by doing constructions *within* extension. We regain explicitly what we have already read-in through intuition.

[3] According to De Risi (2007) Grassmann's algebra should be seen as the more direct descendent of Leibniz's characteristic in the 1679 Huygens letter and the late, c. 1714, *Initia Rerum Mathematicarum Metaphysica*. Grassmann himself certainly believed he had perfected the geometric characteristic along the scientific and philosophical lines shown by Leibniz.