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Hilbert's axiomatic method and Carnap's general axiomatics

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ABSTRACT

This paper compares the axiomatic method of David Hilbert and his school with Rudolf Carnap's general axiomatics that was developed in the late 1920s, and that influenced his understanding of logic of science throughout the 1930s, when his logical pluralism developed. The distinct perspectives become visible most clearly in how Richard Baldus, along the lines of Hilbert, and Carnap and Friedrich Bachmann analyzed the axiom system of Hilbert's *Foundations of Geometry*—the paradigmatic example for the axiomatization of science. Whereas Hilbert's axiomatic method started from a local analysis of individual axiom systems in which the foundations of mathematics as a whole entered only when establishing the system's consistency, Carnap and his Vienna Circle colleague Hans Hahn instead advocated a global analysis of axiom systems in general. A primary goal was to evade, or formalize *ex post*, mathematicians' 'material' talk about axiom systems for such talk was held to be error-prone and susceptible to metaphysics.

The present paper studies the relation between the axiomatic method, advocated and practiced by David Hilbert and his school since *Foundations of Geometry* (1899), and Rudolf Carnap's general axiomatics. Although Hilbert's work was not without predecessors, he and his school used the axiomatic method more systematically and more broadly than ever before, applying it not only to the various fields of mathematics but also to the empirical sciences. Moreover, Hilbert often reflected about the philosophical significance of the axiomatic method in the way characteristic of the scientist-philosophers of the day. Carnap was early on attracted by an "axiomatic approach to both mathematics and physics, as championed by Hilbert" (Reck, 2007, 179)—his first dissertation project read "Axiomatic Foundations of Kinematics"—and tried to reconcile it with the logicist approach that he had learned from Frege and Russell. The major outcome of Carnap's efforts was the unpublished manuscript *Untersuchungen zur allgemeinen Axiomatik* (Investigations on General Axiomatics, 2000), written in 1928–9, and a joint paper with the mathematician Friedrich Bachmann (Carnap & Bachmann, 1936), both of which have recently attracted considerable scholarly attention (Hintikka, 1991; Loeb, 2014a, 2014b; Schiemer, 2012a, 2012b, 2013).

While these scholars investigate the philosophical relevance of Carnap's works from the perspective of historical and systematic issues in model theory and semantics, the present paper starts from the actual scientific practice that Carnap was addressing and that had motivated his aborted dissertation project. At the beginning of Carnap's scientific career, Hilbert's *Foundations of Geometry* had become the paradigmatic example for a large number of axiomatizations in mathematics and in the sciences, above all in physics. These axiomatizations—or, as it was sometimes called, the field of axiomatics—represented a most suitable point of entry for a philosophy based on the formal methods developed in the sciences themselves. Axiomatization is accordingly an important early example of what is today understood as formal epistemology.[1]

In *Untersuchungen,* Carnap defines general axiomatics as "the theory of general, logical–formal properties of axiom systems and

[1] Spelling out this point would require a much broader study that discusses, both historically and systematically, the impacts of mathematical physics and game theory—arising from within the Hilbert school—on present-day formal epistemology and relates them to other fields, such as the mathematization of psychology and inductive logic. For a case study that places John von Neumann's (1932) influential axiomatization of quantum mechanics within the context of the Hilbert school, cf. Stöltzner (2001).

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the relations between axiom systems, in distinction to 'special axiomatics' that deals with certain axiom systems (e.g., an axiom system of Euclidean geometry, of set theory, etc.)" (2000, 59).[2] In this paper I argue that although Carnap's specific proposal in *Untersuchungen* fell short, the concept of general axiomatics continued to influence his understanding of geometry and logic of science still in the 1930s. The principled reason was that Hilbert's axiomatic method was not just a philosophical program, but primarily a scientific practice widespread in the 1920s and 1930s ('special axiomatics'), whose philosophical analysis required a suitably general approach. Thus, I take a significant part of Carnap's work during the 1930s, including his references to the logical problems in geometry, as contributions to the field of axiomatics standing alongside the work of others in applying and analyzing the axiomatic method.

A major reason for this continuity in Hilbert and Carnap is that the 'foundational' ambitions involved in axiomatizing a physical or mathematical theory depend in a significant way on, but are not identical with, how one construes the foundations of concrete individual axiom systems. Unfortunately, Hilbert's extensive use of 'foundations' (*Grundlagen*) effectively blurs an important distinction here even though the two senses were well distinguished within the Göttingen practice of the axiomatic method. By setting up a mathematical axiom system for a physical, or another mathematical, theory the Hilbert school was laying its *mathematical* foundations; through the logical (or meta-mathematical) justification of such an axiom system, it investigated the *logical* (or metamathematical) foundations. *Foundations of Geometry* (1899) had laid the mathematical foundations of geometry in such a way as to require a suitable logical foundation of arithmetic because all geometrical concepts were interpreted by reducing them to appropriately defined number fields, which in turn became the object of logical (or metamathematical) analysis. The latter was pursued in works titled "(Logical) Foundations of Mathematics" (Hilbert 1923, 1928; Hilbert & Bernays, 1934) which were increasingly viewed as key to the foundation of all mathematics.

This distinction between two senses of 'foundations' inherent in Hilbert's axiomatic method differs from Carnap's distinction between special and general axiomatics because Carnap did not follow Hilbert's separation of mathematics and logic (or metamathematics). Still, Carnap reconfigured his own understanding of logic, and accordingly his own understanding of a logical interpretation of mathematics. While *Untersuchungen* had taken an essentially logicist route and considered mathematics as part of logic proper, presupposing however an uninterpreted 'basic discipline' (*Grunddisziplin*), Carnap (1939) later advocated a much broader conception of logic that embraced all basic mathematical disciplines, among them the analysis of real numbers and infinitesimal calculus. Instead of grounding mathematics upon logic, he simply integrated large parts of it into logic. If one calls Carnap's later position anti-foundationalism, as is sometimes done, such a classification is directed only against the second sense of Hilbert, the logical or metamathematical foundations.

Hilbert's axiomatic method brought major challenges not only for the justification of axiom systems, but also for the, individual and comparative, analysis of their logical properties. "What has become crucial instead [of the traditional emphasis of intuitive certainty and clarity] is the systematic investigation, by increasingly abstract and formal means, of three logical properties of an axiom system: (a) the independence of its axioms; (b) their consistency; and their completeness." (Reck, 2007, 182) Independence meant that an axiom could not be derived from the others, such that is was not dispensable. Consistency was required for the axioms to describe a mathematical object at all. Completeness meant that the system could not be extended by further axioms without running into inconsistencies. Carnap (2000) initially believed that these properties could be satisfactorily codified by some version of general axiomatics in a straightforwardly formal fashion. But in the 1930s, it turned out that the logical analysis of their function and mutual relationships was difficult and full of ambiguities. Consider consistency: after Carnap's (2000) own attempts to reach a unified concept of completeness by proving the equivalence of the three concepts then available[3] turned out to be inadequate, the respective significance of these concepts remained unclear at first. Then Gödel showed that consistency could not be established in an absolute sense along the lines of logicism or metamathematics by availing himself of one notion of completeness (decidability or syntactic completeness). Later Carnap and Bachmann, in 1936, attempted a new analysis of completeness, again within a modified general axiomatics. Although their attempt was not entirely successful, it provided important insights into problems that would subsequently be treated by stronger model-theoretic means.

Moreover, having proven the independence of an axiom from the others, what consequences should the mathematician working in special axiomatics draw for the further study of an axiom system? While Carnap only saw room for pragmatic choices among the available alternatives, the Hilbert school focused on 'deepening the foundations' of the respective axioms, that is, on pursuing the axiomatic analysis until a suitably fundamental concept or structure was found. From the perspective of Carnap and his Vienna Circle colleague and eminent mathematician Hans Hahn, however, the idea that some axioms were deeper than others appeared deeply suspect.

Let me condense the foregoing into the following theses. (i) General axiomatics remained an important topic in Carnap's work until the mid-1930s. This implies, in agreement with current scholarship, that the joint paper with Bachmann was not a mere outlier. (ii) While Carnap and Hahn advocated a global analysis of axiom systems, Hilbert's axiomatic method typically started from a local analysis of individual axiom system. Metamathematics only represented one specific part of Hilbert's axiomatic method. (iii) The difference between the two approaches becomes clearest in how Richard Baldus, along the lines of Hilbert, and differently from Carnap and Friedrich Bachmann, analyzed the axiom system of Hilbert's *Foundations of Geometry* and the completeness axiom. The analysis of the logical and methodological properties of axiom systems revealed various problems. While completeness represented a criterion that was tractable, though not without problems, by formal means, simplicity and fertility were clearly of a pragmatic nature. But the structural criterion that Hilbert discussed under the heading of 'deepening the foundations', from Carnap and Hahn's point of view, involved vague material talk and strides into metaphysics.

This paper is organized as follows. Section 1 discusses Carnap's and Hahn's contributions to the 1929 and 1930 meetings on the "Epistemology of the Exact Sciences." They concern Carnap's early general axiomatics and the broader issue of the applicability of mathematics. Section 2 presents the axiomatic method in the Hilbert School and its critical reception in the Vienna Circle. A major source of disagreement consisted in how to understand 'deepening the foundations' because it was neither formal nor pragmatic. Section 3 compares in detail how Baldus and Carnap and Bachmann

[2] Unless a specific translation is mentioned, all translations from German are mine.

[3] These were monomorphy, non-forkability, and decidability—in Carnap's parlance; see Section 1.