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An impossibility theorem for parameter independent hidden variable theories

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ABSTRACT

Recently, Roger Colbeck and Renato Renner (C&R) have claimed that '[n]o extension of quantum theory can have improved predictive power' (Colbeck & Renner, 2011, 2012b). If correct, this is a spectacular impossibility theorem for hidden variable theories, which is more general than the theorems of Bell (1964) and Leggett (2003). Also, C&R have used their claim in attempt to prove that a system's quantum-mechanical wave function is in a one-to-one correspondence with its 'ontic' state (Colbeck & Renner, 2012a). C&R's claim essentially means that in any hidden variable theory that is compatible with quantum-mechanical predictions, probabilities of measurement outcomes are independent of these hidden variables. This makes such variables otiose. On closer inspection, however, the generality and validity of the claim can be contested. First, it is based on an assumption called 'Freedom of Choice'. As the name suggests, this assumption involves the independence of an experimenter's choice of measurement settings. But in the way C&R define this assumption, a no-signalling condition is surreptitiously presupposed, making the assumption less innocent than it sounds. When using this definition, any hidden variable theory violating parameter independence, such as Bohmian Mechanics, is immediately shown to be incompatible with quantum-mechanical predictions. Also, the argument of C&R is hard to follow and their mathematical derivation contains several gaps, some of which cannot be closed in the way they suggest. We shall show that these gaps can be filled. The issue with the 'Freedom of Choice' assumption can be circumvented by explicitly assuming parameter independence. This makes the result less general, but better founded. We then obtain an impossibility theorem for hidden variable theories satisfying parameter independence only. As stated above, such hidden variable theories are impossible in the sense that any supplemental variables have no bearing on outcome probabilities, and are therefore trivial. So, while quantum mechanics itself satisfies parameter independence, if a variable is added that changes the outcome probabilities, however slightly, parameter independence must be violated.

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1. Introduction

In 1935, Einstein, Podolsky and Rosen famously argued that quantum mechanics is incomplete and that there might be another theory that does provide a complete description of physical reality (Einstein et al., 1935). One class of candidates for such a theory is the class of so-called 'hidden variable theories', which supplement the quantum state with extra variables.[1] Hidden variable theories have indeed been developed, for example the de Broglie–Bohm theory (Bohm, 1952), which is deterministic and complete in Einstein's sense. However, a number of impossibility theorems have been derived, showing that large classes of possible hidden variable theories are incompatible with quantum-mechanical predictions. John Bell proved such incompatibility for local deterministic hidden variable theories (Bell, 1964), as well as for local stochastic hidden variable theories (Bell, 2004), while the incompatibility of 'crypto-nonlocal' theories was proven by Leggett (2003). Still, a large class of hidden variable theories, like the de Broglie–Bohm theory, remains unscathed by these impossibility theorems.

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[1] In the literature, sometimes the term 'hidden variable theory' is understood to refer only to deterministic theories. Instead, we have a very general use of the term in mind: theories that add, in addition to the quantum state, an extra variable to the description of a system.

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The hidden variable theories shown to be incompatible by Bell are theories satisfying a criterion called Factorizability (Fine, 1982), which is equivalent to the conjunction of two locality conditions coined by Shimony (1984): parameter independence (ParInd) and outcome independence (OutInd). Therefore, any hidden variable theory compatible with quantum mechanics violates at least one of these two conditions. In this article we claim something stronger: any hidden variable theory compatible with quantum mechanics violates ParInd, except for 'trivial' hidden variable theories, where the values of the hidden variables have no bearing on measurement outcome probabilities.

This article is based on recent work by Roger Colbeck and Renato Renner (C&R), who have claimed that they have derived an even more general impossibility theorem (Colbeck & Renner, 2011, 2012b). Stating that '[n]o extension of quantum theory can have improved predictive power', they essentially claim that *any* non-trivial hidden variable theory, also if it violates ParInd (like the de Broglie-Bohm theory), is incompatible with quantum-mechanical predictions. Given the wide scope of this claim, this would be a spectacular result, which would to a great extent put constraints on any possible future theory replacing quantum mechanics.

However, C&R's claim crucially hinges on an assumption dubbed 'Freedom of Choice'. As the name suggests, this assumption is meant to be about the freedom of experimenters to choose their measurement settings. From this assumption, C&R derive 'no-signalling', which is essentially equal to ParInd. Nevertheless, when inspecting the way 'Freedom of Choice' is defined, it becomes apparent that ParInd is in fact part of this assumption. Most criticism of C&R's work focuses on this issue (Colbeck & Renner, 2013; Ghirardi & Romano, 2013a, 2013; Leifer, 2014; Landsman, 2015). We agree with the criticism: C&R's 'Freedom of Choice' assumption is much stronger than its name suggests. Therefore, while the impression is given that any hidden variable theory with free experimenters is shown to be incompatible with QM, in fact their result applies to a smaller class of hidden variable theories: those satisfying ParInd. The de Broglie–Bohm theory, which violates ParInd, is therefore not shown to be incompatible after all.

If the above issue was the only problem with C&R's work, the result of the present article could easily be achieved by adding ParInd as an explicit assumption. The theorem would then still be an interesting impossibility theorem, being more general than the theorems of Bell and Leggett. However, there are more shortcomings in the work of C&R. First, it is hard to understand, even for experts. Scarani (2013) says:

> 'Beyond the case of the maximally entangled state, which had been settled in a previous paper, they prove something that I honestly have not fully understood. Indeed, so many other colleagues have misunderstood this work, that the authors prepared a page of FAQs [Colbeck, 2010] (extremely rare for a scientific paper) and a later, clearer version [Colbeck and Renner, 2012b].'

The case of the maximally entangled state that Scarani refers to corresponds to the triviality claim of C&R restricted to local measurements on a Bell state. This result consists of the statement that not only the quantum-mechanical outcome probabilities, but also the outcome probabilities in any hidden variable theory equal 1/2 (in the present paper, this result is presented in Section 4). Some authors, for example Di Lorenzo (2012), appear to have understood C&R as deriving only this result, which, as Scarani alludes to, had been derived before. Actually, for C&R this is only the first step in proving the more general theorem that probabilities in hidden variable theories are always equal to the quantum-mechanical probabilities.

More importantly, C&R's derivation contains gaps, of which some are allegedly filled in other publications, while others remain. One example is their careless handling of limits: in more than one occasion results are derived that only hold approximately, which are then used as if they hold exactly.[2]

Because of these shortcomings, at present no acceptable deduction of the impossibility theorem for hidden variable theories satisfying ParInd exists in the literature. In this article we attempt to repair the shortcomings of C&R's derivation in order to establish such a deduction. A step that is not explicitly mentioned by C&R, involving the relation between measurements on entangled states and measurements on non-entangled states, is formulated explicitly. Furthermore, we give a deduction that is mathematically acceptable. We emphasize that this does not consist of simply filling some gaps. For some parts of the deduction to succeed, an entirely new strategy has to be constructed, or so we claim. This is especially the case when taking proper care of all the limits used in the proof. Also, some parts of the deduction can, in our opinion, be considerably simplified, especially the first steps. For these reasons, in this article we do not merely point out all the shortcomings in the original derivation; rather, we construct a new version of it.

C&R have also used their claim in an attempt to answer the question whether the quantum-mechanical wave function is 'ontic' or 'epistemic'. Since the appearance of the Pusey–Barret–Rudulph (PBR) theorem (Pusey et al., 2012), this is a hotly debated topic. On the basis of their claim, C&R argue not only that the wave function is ontic, but also that it is in a one-to-one correspondence with its ontic state (Colbeck & Renner, 2012a). In the Discussion (Section 9), we shall consider what remains of this ψ-ontology result if C&R's claim is replaced by the weaker result deduced in this article.

The result will be deduced in several steps. The first steps are quite simple and correspond to results that existed already before the work of C&R. However, we believe that even for those whom are already familiar with this result, these steps are still of value since they are considerably simplified, only using a triangle inequality and a simple inequality from probability theory. The final steps require more mathematics and may be harder to follow. Most of the mathematics is relegated to appendices, so as not to distract the reader from the central line of reasoning. If the reader want to shorten the reading time, the best section to skip might be Section 7, because the extent of the generalization (from states with coefficients that are square roots of rational numbers to any coefficients) is relatively small compared to the amount of mathematics needed. It is however a necessary part for deriving the full theoretical result. In the Discussion, I shall mention the most important differences between our deduction and that of C&R.

2. Notation

Quantum-mechanical systems are referred to by the symbols A, B, A', B', etc. To denote composite systems, the symbols for the subsystems are combined, for example AB and $AA'A''$. The Hilbert space of system A is denoted by $\mathcal{H}_A$, a state as $|\psi\rangle_A$ and an operator on $\mathcal{H}_A$ as U^A. For notational convenience, the subscript attached to a state may be omitted when no confusion is possible, especially when large composite systems like $AA'A''BB'B''$ are involved. The symbol $\otimes$ for taking tensor products is also often omitted, and we freely change the order of states when combining systems, so that

[2] Section 9 contains a more detailed treatment of these gaps.