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Characterizing common cause closedness of quantum probability theories

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ABSTRACT

We prove new results on common cause closedness of quantum probability spaces, where by a quantum probability space is meant the projection lattice of a non-commutative von Neumann algebra together with a countably additive probability measure on the lattice. Common cause closedness is the feature that for every correlation between a pair of commuting projections there exists in the lattice a third projection commuting with both of the correlated projections and which is a Reichenbachian common cause of the correlation. The main result we prove is that a quantum probability space is common cause closed if and only if it has at most one measure theoretic atom. This result improves earlier ones published in Gyenis and Rédei (2014). The result is discussed from the perspective of status of the Common Cause Principle. Open problems on common cause closedness of general probability spaces $(\mathcal{L}, \phi)$ are formulated, where $\mathcal{L}$ is an orthomodular bounded lattice and ϕ is a probability measure on $\mathcal{L}$.

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1. The main result

In this paper we prove new results on common cause closedness of quantum probability spaces. By a quantum probability space is meant here the projection lattice of a non-commutative von Neumann algebra together with a countably additive probability measure on the lattice. Common cause closedness is the feature that for every correlation between a pair of commuting projections there exists in the lattice a third projection commuting with both of the correlated projections and which is a Reichenbachian common cause of the correlation.

The main result we prove is that a quantum probability space is common cause closed if and only if it has at most one measure theoretic atom. Since classical, Kolmogorovian probability spaces were proved in Gyenis and Rédei (2011) to be common cause closed if and only if they contained at most one measure theoretic atom, and since classical probability spaces also can be regarded as projection lattices of *commutative* von Neumann algebras, this result gives a complete characterization of common cause closedness of probability spaces in the category of von Neumann algebras. Previous results on common cause closedness of quantum probability spaces had to assume an additional, somewhat artificial and not very transparent feature of the quantum probability measure under which the quantum probability space could be proved to be common cause closed (Gyenis & Rédei, 2014). With the removal of that condition it becomes visible that exactly the same type of measure theoretical structure is responsible for the common cause closedness (or lack of it) of classical and quantum probability spaces.

The broader context in which we give our proofs is the problem of characterization of common cause closedness of general probability spaces $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is an orthocomplemented, orthomodular, bounded σ-lattice and ϕ is a countably additive general probability measure on $\mathcal{L}$. (Classical and quantum probability spaces are obviously special examples of abstract probability spaces.) However, little is known about the problem of common cause closedness in this generality. A sufficient condition for common cause closedness of general probability theories is known (Proposition 3.10 in Gyenis & Rédei, 2014, recalled here as Proposition 5) but the condition is exactly the not very natural

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one that could be eliminated both in classical and in quantum probability spaces, and one would like to know whether it also can be eliminated (or replaced by a more natural one) in general probability theories (Problem 15). It also is unknown whether the condition which is necessary for common cause closedness of quantum probability spaces is necessary for the common cause closedness of general probability theories as well (Problem 16). Further open questions and possible directions of investigation will be indicated in Section 7.

The conceptual-philosophical significance of common cause closed probability spaces is that they display a particular form of causal completeness: these theories themselves can explain, exclusively in terms of common causes that they contain, *all* the correlations they predict; hence these theories comply in an extreme manner with the Common Cause Principle. Probabilistic physical theories in which the probability space is measure theoretically purely non-atomic are therefore good candidates for being a confirming evidence for the Common Cause Principle. Section 6 discusses this foundational-philosophical significance of the presented results in the context of the more general problem of assessing the status of the Common Cause Principle.

The other sections of the paper are organized as follows. Section 2 fixes some notation and recalls some basic definitions in lattice theory. In Section 3 the notion of common cause in general probability theories is defined. In Section 4 it is shown that for a probability space, classical or quantum, to be common cause closed it is *sufficient* that they have at most one measure theoretic atom (Propositions 7 and 9). Section 5 proves that this condition is also *necessary*, both in case of classical probability spaces (Proposition 12) and in quantum probability spaces (Proposition 14). Section 7 formulates some problems that are open at this time.

2. General probability spaces – definitions and notations

Throughout the paper $\mathcal{L}$ denotes an orthocomplemented lattice with lattice operations $\vee$, $\wedge$ and orthocomplementation $\perp$. The lattice $\mathcal{L}$ is called orthomodular if, for any $A, B \in \mathcal{L}$ such that $A \leq B$, we have

$$B = A \vee (B \wedge A^{\perp}) \tag{1}$$

The lattice $\mathcal{L}$ is called a Boolean algebra if it is distributive, i.e. if for any $A, B, C \in \mathcal{L}$ we have

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \tag{2}$$

It is clear that a Boolean algebra is an orthomodular lattice. Other examples of orthomodular lattices are the lattices of projections of a von Neumann algebra; they are called von Neumann lattices. The projection lattice of a von Neumann algebra is distributive if and only if the von Neumann algebra is commutative. A basic reference for orthocomplemented lattices is Kalmbach (1983). For a summary of basic facts about von Neumann algebras and von Neumann lattices we refer to Rédei (1998), for the theory of von Neumann algebra our reference is Kadison and Ringrose (1986). The paper (Rédei & Summers, 2007a) gives a concise review of the basics of quantum probability theory.

If, for every subset S of $\mathcal{L}$, the join and the meet of all elements in S exist, then $\mathcal{L}$ is called a complete orthomodular lattice. If the join and meet of all elements of every countable subset S of $\mathcal{L}$ exist in $\mathcal{L}$, then $\mathcal{L}$ is called a σ-lattice. Von Neumann lattices are complete hence σ-complete. In the present paper, it is assumed that lattices are bounded: they have a smallest and a largest element denoted by 0 and I, respectively.

Let $\mathcal{L}$ be a σ-complete orthomodular lattice. Elements A and B are called mutually orthogonal if $A \leq B^{\perp}$. The map $\phi : \mathcal{L} \to [0,1]$ is called a (general) probability measure if $\phi(I) = 1$ and $\phi(A \vee B) = \phi(A) + \phi(B)$ for any mutual orthogonal elements A and B. A probability measure ϕ is called a σ-additive probability measure if for any countable, mutually orthogonal elements $\{A_i \mid i \in \mathbb{N}\}$, we have

$$\phi(\vee_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \phi(A_i) \tag{3}$$

Next we consider atoms. There are two types of atoms: algebraic and measure theoretic. An element $A \in \mathcal{L}$ is called an algebraic atom if $A > 0$ and, for any $B \leq A$ we have $B = A$ or $B = 0$. The other type of atom depends on a probability measure on $\mathcal{L}$. Let ϕ be a probability measure on $\mathcal{L}$. An element $A \in \mathcal{L}$ is called a ϕ-atom if $\phi(A) > 0$ and, for any $B \leq A$ we have $\phi(B) = \phi(A)$ or $\phi(B) = 0$.

A probability measure ϕ on $\mathcal{L}$ is called

- purely atomic if for any $A \in \mathcal{L}$ with $\phi(A) > 0$ there exists a ϕ-atom $B \in \mathcal{L}$ such that $B \leq A$,
- purely nonatomic if for any $A \in \mathcal{L}$ with $\phi(A) > 0$ there exists an element $B \in \mathcal{L}$ such that $B < A$ and $0 < \phi(B) < \phi(A)$.

If $\phi(A) = 0$ implies $A = 0$ for any $A \in \mathcal{L}$, then ϕ is called faithful. Roughly speaking, this condition means that the elements whose probabilities are zero are ignored because such elements are identified with the zero element. According to the following lemma, we can identify a ϕ-atom with an atom in the case where ϕ is faithful. Since we will deal with faithful measures in the paper, algebraic and measure theoretic atoms can be identified and this will be done implicitly in this paper.

Lemma 1. *Let ϕ be a faithful probability measure on $\mathcal{L}$. A is a ϕ-atom if and only if A is an atom.*

Proof. Let A be a ϕ-atom. For any B such that $B \leq A$, $\phi(B) = \phi(A)$ or $\phi(B) = 0$. $\phi(B) = 0$ implies $B = 0$ because ϕ is faithful. $\phi(B) = \phi(A)$ implies $\phi(A^{\perp} \wedge B) = 0$, so that $A = B \vee (A^{\perp} \wedge B) = B$. This means that A is an atom. It is trivial that A is a ϕ-atom if A is an atom.□

We say that two elements A and B in an orthomodular lattice $\mathcal{L}$ are compatible if

$$A = (A \wedge B) \vee (A \wedge B^{\perp}) \tag{4}$$

It can be shown in Kalmbach (1983, Theorem 3.2) that (4) holds if and only if

$$B = (B \wedge A) \vee (B \wedge A^{\perp}) \tag{5}$$

In other words, the compatibility relation is symmetric. If $\mathcal{L}$ is a Boolean algebra, any two elements in $\mathcal{L}$ are compatible.

3. Definition of common cause closedness

In order to investigate common cause closedness in an orthomodular lattice, we must re-define both the concept of correlation and the notion of common cause of the correlation with which we define common cause closedness. The reason why these concepts have to be re-defined explicitly is that Reichenbach's original definition was given in terms of classical probability spaces (Reichenbach, 1956, Section 19), and in such probability spaces all random events are compatible. In the lattice $\mathcal{L}$ of a general probability space there exist elements however which are not compatible. Hence it must be stipulated explicitly whether we allow (i) incompatible elements to be correlated and (ii) the common causes of correlations to be incompatible with the elements in the correlation. We take a conservative route by disallowing such cases: