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On broken symmetries and classical systems

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ABSTRACT

Baker (2011) argues that broken symmetries pose a number of puzzles for the interpretation of quantum theories—puzzles which he claims do not arise in classical theories. I provide examples of classical cases of symmetry breaking and show that they have precisely the same features that Baker finds puzzling in quantum theories. To the extent that Baker is correct that the classical cases pose no puzzles, the features of the quantum case that Baker highlights should not be puzzling either.

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1. Introduction

Baker (2011) argues that broken symmetries play a crucial role in guiding our interpretation of quantum field theory. Baker and others (Baker & Halvorson, 2013; Earman, 2003; Ruetsche, 2011) approach the topic of symmetry breaking through the algebraic formalism, in which one begins by representing physical quantities as elements of an abstract C*-algebra, and then one looks for concrete representations of that algebra in the bounded operators on some Hilbert space. Whenever a symmetry is broken, there are multiple *unitarily inequivalent representations* of the abstract algebra. According to Baker, the appearance of unitarily inequivalent representations in quantum cases of symmetry breaking gives rise to a number of puzzles—puzzles that don't appear in the classical case. He believes we must solve these puzzles to arrive at an adequate interpretation of quantum field theory.[1]

This paper proposes to pull the discussion back from the context of quantum physics—the interpretation of which is extremely controversial—to that of classical physics—which is at the very least better understood. I will analyze classical cases of symmetry breaking in order to compare their features with the quantum cases. The basic strategy of this paper is to compare the mathematical features of symmetry breaking in quantum and classical theories by putting both kinds of theories on common mathematical ground. We don't have to look far to do so: it just so happens that one can use the very same algebraic formalism previously mentioned to describe classical theories as well as quantum ones.[2] Given this algebraic reformulation of classical physics one can compare in detail classical cases of symmetry breaking with quantum cases.

In this paper, I will show that classical cases of symmetry breaking—when translated into the algebraic formalism—give rise to unitarily inequivalent representations. I will illustrate this with two simple explicit examples of classical symmetry breaking: the classical real scalar field and the classical spin chain. All parties agree that the classical cases of symmetry breaking pose no interpretive puzzles.[3] To the extent that this is correct, it means that the presence of unitarily inequivalent representations is *not in*

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[1] As we will see, Baker ultimately believes that these puzzles can be solved, so in that regard my conclusions may not differ substantially from where Baker ends up. Still, we take very different routes to that position, and I think there is insight to be gained from studying the examples here.

[2] See, e.g., Summers & Werner (1987, p. 2441), Brunetti, Fredenhagen, & Ribeiro (2012), and Landsman (1998).

[3] For the main argument of this paper, it does not matter what reasons—mathematical, metaphysical, or otherwise—one has for believing that classical cases of symmetry breaking pose no interpretive puzzles. Baker (2011, p. 132) provides arguments that the classical case is well understood, and I will provide some remarks from a somewhat different perspective (Section 4.2) to suggest this is correct. If one rejects Baker's reasons or my own, then one can substitute whatever reasons he or she prefers and my argument remains intact. My central claim in this paper is only that the very same feature of unitarily inequivalent representations that Baker finds puzzling in quantum theories appears again in my classical examples.

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itself puzzling. This is not to say that there is nothing interesting about broken symmetries. Rather, my claim is that if there is something philosophically or physically interesting to be learned from broken symmetries, it must involve features beyond the mere presence of unitarily inequivalent representations.

2. Preliminaries

In algebraic quantum theories[4] the physical quantities of a system are represented by the self-adjoint elements of an abstract C*-algebra $\mathfrak{A}$. A *state* on $\mathfrak{A}$ is given by a positive, normalized, linear functional $\omega: \mathfrak{A} \to \mathbb{C}$. A state ω has the following initial interpretation: for each self-adjoint $A \in \mathfrak{A}$, $\omega(A)$ corresponds to the expectation value of A in the state ω. A state ω is *pure* if whenever $\omega = a_1\omega_1 + a_2\omega_2$ for states ω_1, ω_2, it follows that $\omega_1 = \omega_2 = \omega$. Otherwise it is called *mixed*.

Importantly, this abstract algebraic formalism translates back into the familiar Hilbert space theory once we are given a state. A *representation* of a C*-algebra $\mathfrak{A}$ is a pair $(\pi, \mathcal{H})$, where $\pi: \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism into the bounded linear operators on some Hilbert space $\mathcal{H}$. A representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ is *irreducible* if the only subspaces left invariant under the action of $\pi(\mathfrak{A})$ are $\{0\}$ and $\mathcal{H}$. Otherwise it is called *reducible*. One of the most fundamental results in the theory of C^*-algebras, known as the GNS theorem, asserts that for each state ω on $\mathfrak{A}$, there exists a representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathfrak{A}$, known as the *GNS representation for* ω, and a (cyclic) vector $\Omega_\omega \in \mathcal{H}_\omega$ such that for all $A \in \mathfrak{A}$,

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$$

One may find representations of $\mathfrak{A}$ on different Hilbert spaces, and in this case one wants to know when these can be understood as "the same representation". This notion of "sameness" is given by the concept of unitary equivalence:[5] two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are *unitarily equivalent* if there is a unitary mapping $U: \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines the representations, i.e. for each $A \in \mathfrak{A}$,

$$U\pi_1(A) = \pi_2(A)U$$

The specified unitary mapping U sets up a way of translating between density operator states on $\mathcal{H}_1$ and density operator states on $\mathcal{H}_2$, and between observables in $\mathcal{B}(\mathcal{H}_1)$ and observables in $\mathcal{B}(\mathcal{H}_2)$. The GNS representation for a state ω is unique in the sense that any other representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ containing a cyclic vector corresponding to ω is unitarily equivalent to $(\pi_\omega, \mathcal{H}_\omega)$.

A *ground state* is a state of lowest energy. While there are many ways to determine the ground state of a quantum system in the algebraic framework (see Bratteli & Robinson, 1996, pp. 97–98), these all rely on the fact that the Hamiltonian of the system is the generator *in the algebraic sense* of the dynamics. While the Hamiltonian of a classical system can also be understood to generate the dynamical evolution in a geometrical sense, it is important to note that the relation of the Hamiltonian to the dynamics is different in classical theories and quantum theories. As such, the definition of ground state employed in quantum theory will not apply to our discussion of classical physics later on. Instead, we will take a ground state to be (following standard practice in classical physics) a minimum of the Hamiltonian, understood as a scalar function on phase space.

A general *symmetry* is represented in the algebraic framework by an automorphism α of the algebra of observables $\mathfrak{A}$.[6] A symmetry acts on states by the transformation $\omega \mapsto \omega \circ \alpha^{-1}$. A symmetry α is *broken* just in case there is some ground state ω which is not invariant under α, i.e. $\omega \neq \omega \circ \alpha^{-1}$. When a symmetry α is broken for a ground state ω in a model of quantum field theory or quantum statistical mechanics, the GNS representations for ω and $\omega \circ \alpha^{-1}$ are unitarily inequivalent (Baker & Halvorson, 2013; Earman, 2003). To see why, simply notice (Halvorson, 2006, Section 2.2) that each Hilbert space representation can have at most one ground state as a vector state. So if $\omega \circ \alpha^{-1}$ is a distinct ground state from ω, then $\omega \circ \alpha^{-1}$ can only be a vector state—as it must be in its own GNS representation—in a distinct (i.e. unitarily inequivalent) Hilbert space representation from the GNS representation of ω. Baker (2011) argues that the presence of unitarily inequivalent representations when a symmetry is broken leads to a number of puzzles, which we turn to now.

3. Puzzles

3.1. Wigner unitary

Baker and Halvorson (2013) argue that there is a prima facie puzzle to understanding how the GNS representations of two symmetry related states can be unitarily inequivalent. Let $(\mathcal{H}_\omega, \pi_\omega)$ be the GNS representation of a C^* algebra $\mathfrak{A}$ for a state ω and let $(\mathcal{H}_{\omega'}, \pi_{\omega'})$ be the GNS representation of $\mathfrak{A}$ for the symmetry transformed state $\omega' = \omega \circ \alpha^{-1}$, where α is a symmetry of $\mathfrak{A}$. Let us suppose that the symmetry α is broken by ω so that $\omega \neq \omega'$. The puzzle arises, they claim, because the symmetry α gives rise to a transformation from vectors in $\mathcal{H}_\omega$ to vectors in $\mathcal{H}_{\omega'}$ which preserves all inner products. It follows from Wigner's theorem that this transformation is given by a unitary operator $W: \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$ between the two Hilbert spaces, and hence the symmetry is implemented by a unitary operator. Given the guaranteed existence of this unitary operator, how could it possibly be that the GNS representations of $\mathfrak{A}$ for ω and ω' are unitarily inequivalent?

To see why Baker and Halvorson find this puzzling, one simply has to note that the 'Wigner unitary' W has many nice properties that make it act like a symmetry on the algebra of observables $\mathfrak{A}$. Namely, Baker and Halvorson show in their 'Wigner representation theorem' that W satisfies

$$W\pi_\omega(\mathfrak{A}) = \pi_{\omega'}(\mathfrak{A})W \tag{1}$$

which means that W can be thought of as mapping the observables in one representation onto the observables in the other, on the whole. And furthermore, for every $A \in \mathfrak{A}$,

$$W\pi_\omega(\alpha^{-1}(A)) = \pi_{\omega'}(A)W \tag{2}$$

which means that W maps symmetry related observables in the different representations to each other. More specifically, W maps the representation of $\alpha^{-1}(A)$ in the GNS representation of ω to the representation of A in the GNS representation of $\omega' = \omega \circ \alpha^{-1}$. How can W, which seems to implement the symmetry as a unitary operator, fail to be a unitary equivalence?

Baker and Halvorson resolve this puzzle by showing that this 'Wigner unitary' W is not a unitary equivalence when $\omega \neq \omega'$. Even

[4] For more on operator algebras, see Kadison & Ringrose (1997), Sakai (1971), and Landsman (1998). For more on the algebraic formalism and axioms of algebraic quantum theory, see Haag & Kastler (1964), Bratteli & Robinson (1987, 1996) and Emch (1972). For philosophical introductions, see Halvorson (2006) and Ruetsche (2011, Chap. 4).

[5] See Ruetsche (2011, Chap. 2.2) and Clifton & Halvorson (2001, Sections 2.2–2.3) for more on unitary equivalence as a notion of "sameness of representations."

[6] One might want to put further restrictions on which automorphisms count as symmetries of the theory, perhaps by looking only at dynamical symmetries, i.e. ones that commute with the dynamics (Baker, 2011, footnote 1). The symmetries considered in Section 4 are all dynamical symmetries by virtue of being induced by symmetries of a Lagrangian or Hamiltonian.