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‘Shut up and contemplate!’: Lucien Hardy's reasonable axioms for quantum theory

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ABSTRACT

Since the beginning of quantum mechanics, attempts were made to derive it from simple natural axioms or assumptions. These reconstructions suffered from various defects, including the questionable naturalness or the overabundance of the axioms, the mathematical difficulty of the derivation, and the inclusion of a wider range of theories than just quantum mechanics. Recently, in 2001, Lucien Hardy propounded “five reasonable axioms” that seem to elude such criticism. The present paper purports to give a simplified version of this new foundation, to discuss Hardy's original version and subsequent variants by others authors, and to investigate the nature of the relevant axioms in light of their possible connection with correspondence arguments.

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Thinking about foundations pays off in the long run. David Mermin once summarized a popular attitude towards quantum theory as “Shut up and calculate!”. We suggest an alternative slogan: “Shut up and contemplate!” (Hardy & Spekkens, 2002, p. 4).

Quantum mechanics is most commonly taught as an operator algebra over a Hilbert space of states, with a few interpretation rules. In this abrupt approach, the mathematical premises lack intuitive grounding, and the success of the theory is a wonder. One would naturally wish to base the theory on assumptions expressible in more elementary and more physical language, and ideally to show the necessity of these axioms. The Hilbert-space structure would then become a consequence of more natural assumptions, and we would understand why nature requires this esoteric mathematical structure.[1]

Reformulations of this kind have existed since the beginning of quantum mechanics. The first one is the quantum logic initiated by John von Neumann and Garrett Birkhoff in 1936, and extended in the late 1950s in Geneva by Constantin Piron and Josef Jauch. In this approach, any empirical observation is reduced to a set of Yes–No questions on the system. These questions or propositions are partially ordered by a relation of implication, which induces a lattice structure of a special kind. Unlike the lattice of Boolean logic, this lattice is not distributive; but it enjoys other properties (modularity, orthocomplementation, and irreducibility in the case of finite dimension) that make it isomorphic to the set of subspaces of a generalized Hilbert space. This is the so-called representation theorem. Quantum logic seduces one by the very basic character of its premises and by its rigorous axiomatics, although it has a few defects: The physical justification of some of the axioms is not so evident; the representation theorem is not easy to prove; the generalized Hilbert space structure is more general than required by quantum mechanics, and there are difficulties in describing subsystems of a physical system.[2]

Quantum logic was not the only attempt to base quantum mechanics on natural axioms. In an influential attempt of 1957, the Harvard mathematician George Mackey defined states and observables through probabilistic axioms about observed values. Among other old axiomatics, the most noticeable is perhaps Günther Ludwig's, which starts with formal characterizations of

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[1] On the philosophical advantages of such reconstructions, see Grinbaum (2007). “Quantum mechanics” is here used to refer to any theory including the basic kinematics of the theory invented in the late 1920s (not the earlier quantum theory), including the unitary evolution of state-representing density matrices in Hilbert space but not necessarily including the canonical commutation rules and the expression of the Hamiltonian.

[2] Birkhoff & von Neumann (1936), Piron (1964). Cf. Dalla, Luisa, Giuntini, & Rédei (2007), Gabbay, Lehmann, & Engesser (2009), Wilce (2009), Darrigol (2014, Section 8.3).

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preparation and registration procedures, and defines states ("ensembles") and observables ("effects") through the statistics of these procedures. While Mackey and Ludwig thus shifted the foundational basis from quantum logic to the structure of a probabilistic state space, they remained tributary of quantum logic in their derivations of the Hilbert-space structure of quantum mechanics. In Ludwig's case, the demands of mathematical rigor and completeness led to overabundant formalism.[3]

Mackey and Ludwig thus failed to improve on the deductive, rational economy of quantum logic. The real turning point in natural quantum foundations was a memoir of 2001 by the British theoretical physicist Lucien Hardy. Hardy changed the axiomatic game by short-circuiting the representation theorems of quantum logic and by instead deriving quantum mechanics "from five reasonable axioms" about probabilistic state space. In his approach, statistical correlation between discrete measurements is the most basic notion, and the consideration of subspaces and composite systems is essential. The states of a system are defined through measurement probability distributions which may be seen as the expression of information content. Hardy's "axioms" are not axioms in a rigorous mathematical sense, but they are easily translated into mathematically precise propositions from which the Hilbert-state structure of quantum mechanics can be derived.[4]

The main purposes of this essay are to present a compact version of Hardy's original argument, to discuss the nature of the basic assumptions, and to give a short critical history of relevant sources. Although the mathematics used by Hardy and his followers tend to be simpler than those of quantum logic, they still involve notions unfamiliar to most physicists. In order to alleviate this inconvenience, this paper begins with an intuitive introduction to some of the main ideas in the context of the simplest known quantum system: a particle with spin one-half (other degrees of freedom being abstracted away). The second section is a mathematically light treatment of the general case of systems for which every measurement has a finite number of discrete outcomes. The third and last section describes the earlier theories on which this treatment draws. These include Hardy's original theory, two significant improvements by Borivoje Dakić and Časlav Brukner and by Lluís Masanes and Markus Müller, and the information-theoretic axiomatics of Giacomo Mauro D'Ariano, Giulio Chiribella, and Paolo Perinotti.[5]

A few preliminary remarks on axioms, naturalness, and correspondence will help grasp the substance of this article. As was already mentioned, Hardy's axioms are not axioms in a strict mathematical sense, ancient Greek or modern. Rather, they are basic assumptions expressed in an informal but conceptually precise manner. They are sharp enough to lead, in an idealized form and with the help of a few easily accepted additional assumptions, to well-defined mathematical statements whose consequences can be pursued rigorously. For adepts of the rigorous axiomatization of physical theories (as in Hilbert's project), it might have been desirable to begin with a set of mathematical axioms inspired from Hardy's assumptions and sufficient to derive the state-space structure of quantum mechanics. This could be done for instance by first defining the state space as a convex domain of $[0,1]^K$ (K being the number of degrees freedom), by introducing affine probability functions on this space (for measurement outcomes), and by enunciating a list of axioms for further determining the state space and the probability functions (for example, there should be a compact group acting transitively on the state space). Or this could be done in an information-theoretic framework as D'Ariano, Chiribella, and Perinotti have done with axioms somewhat remote from Hardy's.

But the gain would hardly compensate the loss of conceptual transparency. What is most needed for the intelligibility of quantum mechanics is not one more rigorous axiomatization in a mathematical sense (we already have von Neumann's axiomatics based on Hilbert spaces, the quantum-logic axioms, Ludwig's axioms, the rigged-Hilbert space axioms, the C*-algebraic axioms, …) but a set of fundamental, intuitively justified assumptions that are sufficiently clear and precise to form the basis of mathematical deductions. The more problematic aspect of such a project is the intuitive justification of the fundamental assumptions, it is not the mathematical deduction from these assumptions. Hardy's followers have found nothing wanting in his deductions. They have only completed them.

Let us focus on the intuitive justification of Hardy's "axioms" or those of his followers. Are these axioms "very reasonable" or "reasonable" (as Hardy puts it)? Or, to put it more strongly, are they necessary? Such judgments are in danger of being overly subjective and they need to be assessed critically. Naturalness may refer to empirical immediacy, to empirical veracity, to mathematical simplicity, or to fittingness in a given conceptual framework. Empirical immediacy stipulates the direct operational significance of an axiom. Hardy achieves it by focusing on measurement results and physical transformations. Empirical veracity further requires the axiom to be a generalization of commonly accepted experimental facts. We would of course be happy if all the axioms met this criterion: we would thus be able to deduce quantum mechanics from a few empirically obvious principles just as we can, for instance, derive thermodynamics from the impossibility of two forms of perpetual motion (with some background knowledge of course). Mathematical simplicity is a less convincing criterion of naturalness as long as there is no philosophical reason to presuppose that nature chooses the simplest mathematical options. Hardy reluctantly appealed to a simplicity axiom, and his followers managed to dispense with it. Most explicitly, Hardy's naturalness means fitness with probability theory. As a probability theory, he tells us, quantum theory is at least as plausible as classical probability theory. His followers say something similar, just replacing probability theory with information theory. In both cases, the fitness is debatable because what is natural from the point of view of probability or information theory need not be natural from a physical point of view.

More broadly but more implicitly, Hardy wanted to base his reconstruction of quantum theory on operational axioms that would be easily acceptable by any well-educated physicist. He did not spell out the rationale of this potential acceptance. This is what I have tried to do. Essentially, I argue that some of the necessary axioms, for instance the discrete character of measurement outcomes (for bounded systems) are operationally meaningful and empirically well-established results, thus meeting the first two criteria of naturalness. For the rest of the axioms, I argue that they result from correspondence arguments.[6]

Famously, correspondence arguments played a crucial role in the historical construction of quantum theory under Niels Bohr's lead. The new atomic theory, Bohr remarked in the 1910s, admits features of discontinuity that contradict the classical theories of mechanics and electrodynamics. Yet experimental results are still expressed by means of relations between quantities defined and measured classically, such as the energy of an atomic level or the

[3] Mackey (1957), Ludwig (1983, 1985). Ludwig's earliest attempts date from the mid-1950s. In its final form, his theory has no less than seventy-six axioms, most of which are there only for mathematical reasons.

[4] Hardy (2001). More recent reconstructions of quantum mechanics by other authors will be discussed or mentioned below.

[5] The contents of this paper partially overlap with Darrigol (2014, Section 8.4).

[6] Hardy and his followers do not explicitly rely on correspondence arguments. I do not know their opinion on such arguments.