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On the CPT theorem

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ABSTRACT

We provide a careful development and rigorous proof of the CPT theorem within the framework of mainstream (Lagrangian) quantum field theory. This is in contrast to the usual rigorous proofs in purely axiomatic frameworks, and non-rigorous proof-sketches in the mainstream approach. We construct the CPT transformation for a general field directly, without appealing to the enumerative classification of representations, and in a manner that is clearly related to the requirements of our proof. Our approach applies equally in Minkowski spacetimes of any dimension at least three, and is in principle neutral between classical and quantum field theories: the quantum CPT theorem has a natural classical analogue. The key mathematical tool is that of complexification; this tool is central to the existing axiomatic proofs, but plays no overt role in the usual mainstream approaches to CPT.

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1. Introduction and motivation

The CPT theorem says, roughly, that every relativistic quantum field theory has a symmetry that simultaneously reverses charge (C), reverses the orientation of space (or 'parity,' P), and reverses the direction of time (T). In this paper we will state and prove a general version of this theorem, proceeding from first principles and explicitly setting out all required assumptions.

Why re-examine a result that is so widely known? The motivation stems from the fact that, as a general rule, the QFT literature splits rather sharply into two sectors. The first, 'mainstream' sector, typified by such standard texts such as Peskin and Schroeder (1995), Itzykson and Zuber (1980), and Weinberg (1995), speaks the language of working particle physicists, but is often rather relaxed about mathematical rigour.[2] The second, 'axiomatic' sector is fully rigorous, but bears a much looser relationship to the QFTs that actually enjoy predictive success; it includes the axiomatic program of Streater and Wightman, and the purely algebraic approach (AQFT) associated with e.g. Araki, Haag and Kastler. This contrast has been a focus of recent discussion in the foundations community, with Fraser (2009, 2011) arguing that because of the lack of rigour in the mainstream approach, the various axiomatic frameworks provide the more appropriate locus for foundational work, while Wallace (2006, 2011) advocates more foundational focus on the mainstream approach for the sake of contact with real physics.

The literature on the CPT theorem is no exception to this general rule. In the papers that first reported the CPT result (e.g. Luders, 1957; Pauli, 1955) and in the standard textbooks mentioned above, the 'theorem' is that Lagrangians (or Hamiltonians) of a certain kind are necessarily invariant under a CPT transformation of the fields. They establish this result via case-by-case calculations for the fields of most physical interest (e.g. vectors or Dirac spinors in $3+1$ spacetime dimensions), and then perhaps refer the reader to e.g. Streater and Wightman (1964) for a more rigorous and 'general' proof.[3] If one follows up these references, one indeed finds a fully rigorous proof of a result called 'The CPT Theorem,' but the relationship of *that* result to the CPT invariance of Lagrangians is obscure; the same remark applies to such AQFT results as that presented in Yngvason and Borchers (2000). The literature contains a gap: there is no rigorous, general proof available of the CPT theorem within the mainstream approach to QFT.

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[2] We will refer to this sector as 'mainstream' simply for lack of a better word. The name 'Lagrangian' is sometimes used, but that would grate in the present context, in which we specifically do not wish to single out Lagrangian as opposed to, say, Hamiltonian methods.

[3] For a more accessible exposition of the ideas behind this axiomatic approach, see Greenberg (2003).

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Driven by the platitude that both mathematical rigour and contact with real physics are highly desirable, this paper aims to fill that gap. We present a rigorous proof using only the basic geometric and group-theoretic facts on which the CPT result essentially depends. Our approach has the following features, which may be of interest even to readers primarily concerned with the axiomatic sector.

(1) We are concerned solely with the symmetries of Lagrangian or Hamiltonian densities, dynamical equations, and similar objects; we say only enough about quantum field theory *per se* to motivate appropriate transformation laws. In fact, our results apply formally to classical relativistic field theories just as well as to quantum ones. We find that the quantum CPT theorem is an instance of a more general result, other instances of which can be seen as classical PT, classical CPT and quantum PT theorems. In standard approaches to the CPT theorem, the relationship between quantum and classical symmetries is left unclear.

(2) We give a geometric construction of CPT transformations for an arbitrary field, based only on how that field transforms under proper orthochronous Lorentz transformations. This construction is clearly related to the requirements of our proof of the CPT theorem, so it is clear *why* an invariance theorem results for these particular transformations. In the existing QFT literature (both mainstream and axiomatic), the CPT transformations tend to be introduced *ad hoc* and case-by-case.

(3) We rely on a few basic geometric properties of the Lorentz group, so that our results are valid for Minkowski space, and, indeed, for any non-Euclidean signature, in dimension at least three. These properties are absent in dimension two and for Galilean spacetimes (for which we show there is no analogous result). The standard approach relies on a detailed classification of the representations and invariants of the four-dimensional Lorentz group, thus obscuring the basic structure and generality of the result.

(4) Our key technique is passage from the real to the complex Lorentz group. This 'complexification' is also the key idea used to prove the CPT theorem of axiomatic QFT, but it plays no overt role in standard approaches to the 'mainstream' CPT theorem. The present paper shows how the mathematical ideas normally used in axiomatic frameworks also apply more directly in the mainstream approach.[4]

The present paper may also be of interest to those seeking a foundational understanding of the *prima facie* mysterious connection between charge conjugation and spacetime symmetries that is embodied in the CPT theorem (cf. Greaves, 2010; Wallace, 2009).

We develop our argument pedagogically, treating first the simpler case of fields taking values in true representations of the Lorentz group (i.e. tensor fields), and later generalising to include properly projective representations (spinor fields). The reader interested only in the broad outline of our results can skip Sections 5–10.

The structure of the paper is as follows. Sections 2–4 lay the conceptual foundations. Section 2 introduces our basic notion of a 'formal field theory,' and explains how it can be used to study the symmetries of classical and quantum field theories. Section 3 explains the distinction between PT and CPT transformations, and the related idea of charge conjugation. Section 4 uses this framework to give a detailed overview of our results.

Sections 5–9 form the technical heart of the paper. Section 5 states and proves a 'classical PT theorem': we show that for classical field theories whose dynamical fields take values exclusively in *true* representations of the Lorentz group (thus excluding spinor fields), proper orthochronous Lorentz invariance entails 'PT invariance.' Section 6 generalises the result of Section 5: we prove a general invariance theorem that has 'tensors-only' versions of the classical PT theorem, the quantum CPT theorem, and classical CPT and quantum PT theorems as corollaries. Of these, the classical PT and quantum CPT theorems are the most interesting, because their premisses are widely accepted.

We next generalise to spinorial field theories. Section 7 lays out the basic facts concerning covers of the proper Lorentz group. Section 8 explains how the most straightforward attempt to generalise our classical tensorial PT theorem to include spinors fails. Section 9, building on this instructive failure, further generalises the results of Section 6 to the spinorial case; this includes the full quantum CPT theorem.

Section 10 examines how our methods apply beyond Minkowski space. We generalise our results to arbitrary non-Euclidean signatures in dimension at least 3. We also point out why our methods fail in various settings where there is provably no analogue of the CPT theorem.

Section 11 is the conclusion, which reviews some of the main conceptual points made along the way. Some mathematical background is presented in Appendix A, to which the reader should refer as necessary. Appendix B relates our treatment of the covering groups of the Lorentz group to the usual approach in terms of Clifford algebras. Detailed proofs are relegated to Appendix C.

2. Field theories and their symmetries

We will state and prove our invariance theorems in a setting of 'formal field theories,' in which the objects of study are formal polynomials that can equally well be interpreted as dynamical equations or as defining Lagrangian or Hamiltonian densities for classical or quantum field theories. The advantage of this framework (over, say, one that takes the objects of study to be spaces of kinematically allowed fields and their automorphisms) is its neutrality between classical and quantum field theories, and between various interpretations of QFTs (as dynamical constraints on operator-valued distributions, formal algorithms for the generation of transition amplitudes, or anything else).

In this section we explain in detail what a formal field theory is, and how they can be used to describe classical and quantum field theories. In particular, we explain how to analyse *spacetime symmetries* of classical and quantum field theories in terms of an analogous notion for formal field theories.

Initially, 'spacetime' M can be any vector space.[5] We must eventually suppose that M has enough structure for us to speak of 'time-reversing' transformations.

2.1. Classical field theories

A *classical field theory* is a set $\mathcal{D} \subset \mathcal{K}$, where the set $\mathcal{K} \equiv C^\infty(M, V)$ of kinematically allowed fields consists of all smooth functions from spacetime to some finite-dimensional real vector space V.[6] $\mathcal{D}$ is the set of dynamically allowed fields. We are mainly interested in theories $\mathcal{D}$ that consist of the solutions to a system of differential equations with constant coefficients – for brevity, we

[4] Complexification *does* play a key role in the treatment of tensors in an illuminating paper reported by Bell (1955); the latter was the original inspiration for the present paper. We make a further comparison to the axiomatic approach in Section 11.8.

[5] As a matter of convenience, we choose an origin for M (thus making it a *vector* space instead of an affine space). When we discuss symmetries, this choice allows us to focus on the Lorentz group rather than the full Poincaré group; it is justified by an implicit assumption that our field theories are, in an appropriate sense, translation invariant.

[6] If the theory 'contains two or more dynamical fields,' as e.g. electromagnetic theory contains the Maxwell–Faraday tensor field $F_{\alpha\beta}$ and the charge-current density vector field J^α, then V will naturally be written as a direct sum of two or more spaces: $V_{EM} := V_F \oplus V_J$. See Example 2.1.