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The implicit function theorem and its substitutes in Poincaré's qualitative theory of differential equations

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ABSTRACT

We analyze the role of the implicit function theorem and some of its substitutes in the work of Henri Poincaré. Special emphasis is given upon his PhD thesis, his first work on the periodic solutions of the three body problem, his memoir crowned by King Oscar II Prize and its development in *Les méthodes nouvelles de la mécanique céleste*, and finally his contributions on the figures of equilibrium of rotating fluid masses.

Résumé: Nous analysons le rôle du théorème des fonctions implicites et de certains substituts dans l'oeuvre de Henri Poincaré. L'accent est mis en particulier sur sa thèse de doctorat, son premier travail sur les solutions périodiques du problème des trois corps, son mémoire couronné par le Prix du Roi Oscar II et son développement dans *Les méthodes nouvelles de la mécanique céleste*, et finalement ses contributions aux figures d'équilibre d'une masse fluide en rotation.

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1. Introduction

The implicit function theorem is one of the most important and versatile tools of mathematics, not only in analysis, but in geometry as well. Although used since the beginning of calculus, its formalization and rigorous proof had to wait for Cauchy (1831) in the analytic case, and to Dini (1878) in the smooth case. Historical information can be found in Krantz-Parks (2002) and Mingari Scarpello-Ritelli (2002).

This theorem and some of its generalizations have played an important role in the work of Henri Poincaré, from his Thesis in 1879 till his work in celestial mechanics. Poincaré independently reinvented what is now called the Weierstrass preparation theorem in order to extend the Cauchy–Kovalewski theorem to some singular cases. In his first work on the periodic solutions of the three body problem, he substituted to the implicit function theorem a topological result which will be later proved to be equivalent to Brouwer fixed point theorem. In his first memoir on the figures of equilibrium of rotating fluid bodies, Poincaré defined the concept of bifurcation points in a series of equilibria. They are essentially the points where the implicit function theorem does not work, and Poincaré introduced topological and analytic tools to prove their existence. Later, and especially in the monographs he devoted to the mentioned problems, the implicit function theorem and some of its consequences became the fundamental tools.

The aim of this paper is to analyze those contributions, and to show that when Poincaré did not take the simplest way, which is often the case for pioneers, his detours were more than worthwhile and the sophisticated tools he invented to solve local problems became, in the hands of other mathematicians, fundamental for the study of the corresponding global problems.

The scientific work of Poincaré has been recently analyzed in a nice and detailed way in the remarkable books of Gray (2012) and of Verhulst (2012). For Poincaré's work on the three body problem, the reference remains Barrow-Green's (1997) monograph. Those books can be usefully consulted for a more systematic and complete description of the memoirs and monographs considered here.

In this paper, a “thematic” or “transversal” viewpoint is emphasized more than a systematic one. We believe that such a viewpoint may be useful in understanding and analyzing Poincaré's mathematics, because of his exceptional talent in using a definite tool in very different areas of mathematics. Such a viewpoint has already been developed in Mawhin (2000), where, instead of implicit function techniques, Kronecker's index had been emphasized. Other mathematical tools could be considered as well, like non-Euclidean geometry, group theory, calculus of variations or anticipations of exterior calculus for example.

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2. Implicit function and preparation theorems

For the reader's convenience, we recall in this section the statements of the main theorems which will be often mentioned in the sequel.

The first version of the implicit function theorem was stated and proved for analytic mappings by Cauchy (1831), and summarized in Cauchy (1841). If $\mathbb{C}^m$ is the cartesian product of m copies of the complex plane $\mathbb{C}$, and $B(r) \subset \mathbb{C}^m$ denotes the open ball of center 0 and radius $r > 0$, the mapping

$$F : B(r_0) \times B(R_0) \subset \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^p$$

is called *analytic* if it is equal on $B(r_0) \times B(R_0)$ to the sum of its Taylor series.

Theorem 1. *If $F : B(r_0) \times B(R_0) \subset \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^p$ is analytic and such that*

$$F(0,0) = 0, \quad \mathrm{Jac}_y F(0,0) \neq 0,$$

then there exist $r_1 \in (0, r_0)$, $R_1 \in (0, R_0)$, and $f : B(r_1) \to B(R_1)$ analytic such that, in $B(r_1) \times B(R_1)$

$$F(x,y) = 0 \Leftrightarrow y = f(x).$$

Recall that the *Jacobian* or *functional determinant* $\mathrm{Jac}_y F(0,0)$ of F with respect to y at $(0,0)$ is the determinant of the complex $(p \times p)-$matrix whose elements are the (complex) partial derivatives $F'_{i,y_j}(0,0)$ $(1 \le i,j \le p)$.

Let $\mathbb{R}^m$ denote the Euclidean space of dimension m and $B(r) \subset \mathbb{R}^m$ the open ball of center 0 and radius $r > 0$. The classical implicit function theorem for mappings of class C^1 can be stated as follows.

Theorem 2. *If $F : B(r_0) \times B(R_0) \subset \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is of class C^1 and such that*

$$F(0,0) = 0, \quad \mathrm{Jac}_y F(0,0) \neq 0,$$

then there exist $r_1 \in (0, r_0)$, $R_1 \in (0, R_0)$, and $f : B(r_1) \to B(R_1)$ of class C^1 such that, in $B(r_1) \times B(R_1)$,

$$F(x,y) = 0 \Leftrightarrow y = f(x),$$

here the Jacobian $\mathrm{Jac}_y F(0,0)$ is the determinant of the real $(p \times p)-$matrix whose elements are the partial derivatives $F'_{i,y_j}(0,0)$ $(1 \le i,j \le p)$. Of course, in Theorems 1 and 2, the centers 0 of the involved balls, chosen for simplicity, can be replaced by any arbitrary point of the corresponding space. Those theorems essentially give conditions under which a graph in the (x,y) space defined by a system of equations $F(x,y) = 0$ can be seen, in the neighborhood of one of its points (x_0, y_0) as the graph of a function $y = f(x)$.

Although implicit functions were used much earlier, the complete statement and proof of Theorem 2 were only given by Dini (1878) in his mimeographed lectures of analysis of 1877–1878, and reproduced in the monographs of Angello Genocchi (written by Genocchi-Peano, 1884) and Jordan (1893).

In the special situation of Theorem 1 with $p = 1$, the following result gives information in cases where the Jacobian vanishes. $F_y^{(k)}$ denotes the kth (complex) partial derivative with respect to y.

Theorem 3. *If $F : B(r_0) \times B(R_0) \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ is analytic and such that*

$$F(0,0) = F'_y(0,0) = \cdots = F_y^{(m-1)}(0,0) = 0, \quad F_y^{(m)}(0,0) \neq 0, \tag{1}$$

then there exist $r_1 \in (0, r_0)$, $R_1 \in (0, R_0)$, $a_0, a_1, \ldots, a_{m-1} : B(r_1) \to \mathbb{C}$ analytic, vanishing at 0, and $G : B(r_1) \times B(R_1) \to B(R_1)$ analytic such that $G(x,y) \neq 0$ on $B(r_1) \times B(R_1)$ and

$$F(x,y) = [a_0(x) + \cdots + a_{m-1}(x) y^{m-1} + y^m] G(x,y) \quad \text{on } B(r_1) \times B(R_1).$$

In other words, the zeros of $F(x, \cdot)$ in a neighborhood of $(0,0)$ are the solutions of the algebraic equation

$$a_0(x) + \cdots + a_{m-1}(x) y^{m-1} + y^m = 0.$$

This result is usually called *Weierstrass preparation theorem*. For $m = 1$, it implies of course Theorem 1 with $p = 1$. As observed by Lindelöf (1905), Cauchy stated and proved it already in 1831 (Cauchy, 1831), and published it in 1841 (Cauchy, 1841). Carl Weierstrass stated and proved it in his Berlin's lectures around 1860, and published it in 1886 (Weierstrass, 1886). Poincaré, as we shall see, stated, proved and published it in 1879.

3. 1879: Sur les propriétés des fonctions définies par les équations aux différences partielles

Poincaré's (1879) thesis, entitled *Sur les propriétés des fonctions définies par les équations aux différences partielles*, and defended in 1879, starts with some "preliminary lemmas". The first one, called by Poincaré "Théorème de Briot-Bouquet" is nothing but Theorem 1. The unusual name given by Poincaré comes from the fact that the reference he gave for this theorem is the famous treatise on elliptic functions of Briot–Bouquet (1875). This may have pleased Bouquet, a member of the jury. Poincaré added that

> this theorem can be seen as a consequence of the theorem of existence of the integral of a differential equation.[1]

Then Poincaré introduced the concept of an *algebroïd function y* from $\mathbb{C}$ to $\mathbb{C}$, namely a function y which, in a neighborhood of $0 \in \mathbb{C}$, is solution of an equation of the form

$$y^m + A_{m-1}(x) y^{m-1} + \cdots + A_1(x) y + A_0(x) = 0,$$

where $m \ge 1$ is an integer and the A_j vanish at 0 and are analytic near 0.

If now $F(x,y) := \sum_{k=0}^{\infty} A_k(x) y^k$, where the analytic functions A_j of $x := (x_1, \ldots, x_n) \in \mathbb{C}^n$ are such that

$$A_0(0) = \cdots = A_{m-1}(0) = 0, \quad A_m(0) \neq 0$$

(which means that F is analytic in the neighborhood of $(0,0)$ and condition (1) holds), Poincaré stated and proved the following two results as Lemmas 2 and 3.

Lemma 1. *There exist m functions y(x) such that, near 0,*

$$F(x, y(x)) = 0 \quad \text{and} \quad \lim_{x \to 0} y(x) = 0.$$

Lemma 2. *The m functions y(x) are algebroïd of degree m.*

So, a third independent author must be added to Cauchy and Weierstrass for essentially proving the preparation theorem.

In the thesis, those results are applied to the obtention of some extensions of Briot-Bouquet theorems for singular ordinary differential equations to Cauchy–Kowalevski's problem for analytic partial differential equations.

[1] ce théorème peut être regardé comme une conséquence du théorème relatif à l'existence de l'intégrale d'une équation différentielle.