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# “The soul of the fact”—Poincaré and proof

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ABSTRACT

Henri Poincaré acquired a reputation in his lifetime for being difficult to read. It was said that he missed out important steps in his arguments, assumed the truth of claims that would be difficult if not impossible to prove, and in short that he lacked rigour. In the years after his death this view coalesced into an exaggerated claim that his work was simply too vague, and has become a cliché. This paper argues that Poincaré was far from indifferent to rigour, and that what characterises his work is an attempt to convey a particular sense of what it is to understand a topic. Throughout his working life Poincaré was concerned to promote the understanding of many domains of mathematics and physics. This is as apparent in his views about geometry, his conventionalism, and his theory of knowledge, as it is in his work on electricity and optics, on number theory, and function theory. It is one of the ways Poincaré discharged his responsibilities as a scientist, and that it accounts not only for a surprising degree of unity in his work but also gives it its distinctive character—at once profound and elusive.



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## 1. Poincaré and rigour

In his address *L'avenir des mathématiques* to the International Congress of Mathematicians (ICM) of 1908 Poincaré remarked that “Rigour is not everything—but without it there is nothing.”[1] The audience then, and readers of his paper today, may reasonably infer that for Poincaré it was not interesting to dwell on rigour when presenting a proof, but that it was nonetheless essential. For Poincaré, the interesting part was the role a proof plays in understanding a piece of mathematics. But Poincaré cared about rigour, as his correspondence with Fuchs in 1880, his remarks on the Dirichlet problem, and many other comments demonstrate.

The correspondence with Fuchs (see Poincaré, 1921b) began on 29 May 1880 as soon as Poincaré had submitted his essay on differential equations in the complex domain for the prize of the Paris Académie des Sciences. The competition, which was won by Georges Halphen, was the occasion for Poincaré to discover the theory of automorphic functions (see Poincaré, 1997). But in May 1880 Poincaré was still considering the subject from an entirely analytic point of view, and his questions to Fuchs were about the analytic continuation of the quotient of two independent solutions of a linear differential equation. This was a standard research material of the day, and one that Fuchs was the acknowledged expert in, but we have the somewhat comic sight of Poincaré explaining the subtleties of analytical continuation to the older man. What he saw, and Fuchs had missed, was an insight into the global nature of the image defined by the quotient. This derived from Fuchs's immersion in a tradition that emphasised local aspects, such as the nature of singular points of an analytic function, and provided techniques for dealing with them, but was much less well equipped to handle global questions. But nonetheless, it was Poincaré, not Fuchs, who was rigorous and Poincaré - who, through this insistence on rigour, was able to reach the situation where the attention to the behaviour of the inverse of the quotient and the nature of its domain was to lead to the great discovery of the importance of non-Euclidean geometry. And indeed, once Poincaré embarked on the study of the functions he called Fuchsian and Kleinian, he expressed them as summations over the group, just as a doubly infinite series is a sum over the elements of the group of integers, and his proof of the convergence of these series made ingenious use of both the Euclidean and the non-Euclidean geometry of the disc.

Poincaré also had what can be called reluctant criticisms of rigour. Proofs can be too large, he argued in *L'Avenir*, and

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[1] En mathématiques la rigueur n'est pas tout, mais sans elle il n'y a rien (Poincaré, 1908b, p. 932). A full English translation of this paper, which can be read as his response to Hilbert's famous ICM address of 1900, appears in Gray (2012c).

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well-chosen terms, such as ‘uniform convergence’ would encapsulate progress and prevent rigorous proofs from becoming almost incomprehensibly too long. Likewise, calculation should be an irreducible minimum, and never blind. Such proofs, he suggested, while valid, could not be properly understood.

A more substantial objection was that proofs can be wrong in kind, as was the case, he suggested, in potential theory, where they do not mimic the actual processes involved. More-or-less intuitive proofs, he said in an analysis of his own scientific work in a memoir[2] written in 1901, are of the right sort to satisfy a physicist because they leave the mechanism of the phenomena apparent. More rigorous arguments for the existence of solutions depended on convergence arguments but this convergence was usually too slow, and the approximations involved were too complicated, for such approaches to yield effective numerical procedures. The implication is not only that there was a better proof to be found that would speak to both the physicists and the mathematicians. Poincaré was also explicit, in his Poincaré (1890b), that the physicists' understanding was not good enough. He argued instead that one could not be content with the lack of a rigorous proof; analysis itself should be able to solve such problems. Any rigorous solution is, of course, a solution, he went on, and even if crude nonetheless teaches us something.[3] Nor was it needlessly pedantic to seek the rigorous solution of equations that had only been established by approximate methods and that rested on imprecise experimental foundations. In his opinion, no-one could be sure that something less than a rigorous proof was not actually flawed. And how could anyone say that something inadequate for mathematics was yet good enough for physics?—the line was impossible to draw.[4] One could not, as a mathematician, settle for less, and in any case many of these equations had applications not only in physics but also in pure mathematics (for example, he observed, Riemann himself had based his magnificent theory of Abelian functions on his use of Dirichlet's principle).

A further objection to rigour that Poincaré held was that there are occasions when it is not enough. He observed in his Poincaré (1905a) that Hilbert had exposed the formal character of reasoning in geometry, and remarked that even if the same was done for arithmetic and analysis, mathematics could not be reduced to an empty form without mutilating it, and the origin of the axioms would still have to be investigated, however conventional they were taken to be.[5] In *L'Avenir* he remarked that logical correctness is not all.

> When a lengthy calculation has led to a striking result we are not satisfied until we understand why we could have predicted, if not the result in its entirety then at least its most characteristic traits.[6]

And because it is not order per se, but only unexpected order that has a value, the mechanical pursuit of mathematics would be worthless,

> *A machine can take hold of the bare facts, but the soul of the fact will always escape it.*[7]

The problem as Poincaré saw it was: How to proceed? Isolated facts had no appeal for him, but, he suggested, a class of facts held together by analogy brings us into the presence of a law, and as he continued in explicit agreement with Ernst Mach's principle of the economy of thought, “The importance of a fact is measured by the return it gives—that is, by the amount of thought it enables us to economise”.[8] Poincaré argued that the elegance of a good proof reflects an underlying harmony that in turn introduces order and unity and “enables us to obtain a clear comprehension of the whole as well as its parts. But that is also precisely what causes it to give a large return.”[9] The aesthetic response to mathematics - was regarded by Poincaré as a sign of its efficacy, and this pair of ideas then shaped the rest of his address.

## 2. Poincaré on progress in mathematics and physics

Contrary to the clichéd image of Poincaré that prevails in many places, Poincaré was not seduced by flashes of insight. He explicitly commented that these, although convincing at the time, can mislead[10]:

> I have spoken of the feeling of absolute certainty that accompanies inspiration; in the cases cited the feeling was not deceptive, and it is often so; but one must guard against believing that this is a rule without exceptions. This feeling often deceives us without being any the less strong, and one only discovers this when one starts to write out a proof. I have observed this fact most often when ideas have come to me in the morning or evening lying in bed in a half-awake state.

As he put it in his address to the Parisian Society of Psychologists in 1908 (see his Poincaré, 1908c), the unconscious provides points of departure for calculations that must be made consciously, but operates by chance. And one must be careful, for the unconscious presents these ideas with a feeling of certainty even when, on rational analysis, they prove to be worthless.

There was, however, an in-built activity of the mind that Poincaré argued on several occasions was capable of providing knowledge, and that was our ability to reason by recurrence, and this allows for the growth of knowledge. And, he asked rhetorically in his Poincaré (1902a), “Who doubts arithmetic?” (Perhaps no-one in 1900, when he made these remarks at the Paris ICM.)

[2] Published as his Poincaré (1921a), see *Oeuvres, 9*, 2.

[3] En outre, toute démonstration rigoureuse de la possibilité d'un problem en est toujours une solution. […] cette solution sera généralement grossière […] cependant elle nous enseignera toujours quelque chose. See Poincaré (1890b), *Oeuvres, 9*, 32.

[4] Quand aura-t-on le droit de dire que telle démonstration, insuffisante pour l'Analyse, est assez rigoureuse pour la Physique? La limite est bien difficile à tracer. See Poincaré (1890b), *Oeuvres, 9*, 32.

[5] […] en réduisant la pensée mathématique à une forme vide, il est certain qu'on la mutile. Admettons même que l'on ait établi que tous les théorèmes peuvent se déduire […] par de simples combinaisons logiques d'un nombre fini des axiomes [des conventions]. La philosophe conserverait le droit de rechercher les origines de ces conventions. In ‘Les mathématiques et la logique’, 1905, reprinted in *Science et méthode*, p. 158 and Poincaré (2001, p. 464).

[6] […] quand un calcul un peu long nous a conduits à quelque résultat simple et frappant, nous ne sommes pas satisfaits tant que nous n'avons pas montré que nous aurions *pu prévoir*, sinon ce résultat tout entier du moins ses traits les plus caractéristiques. In *L'Avenir*, p. 932.

[7] “La machine peut mordre sur le fait brut, l'âme du fait lui échappera toujours”. In *L'Avenir*, p. 932, italics in the original.

[8] L'importance d'un fait se mesure donc à son rendement, c'est-à-dire à la quantité de pensée qu'elle nous permet d'économiser. In *L'Avenir*, p. 931.

[9] Qest-ce qui nous donne en effet dans une solution, dans une démonstration, le sentiment de l'élégance? Cest l'harmonie des diverses parties, leur symétrie, leur heureux balancement; cest en un mot tout ce qui y met de l'ordre, tout ce qui leur donne de l'unité, ce qui nous permet par conséquent dy voir clair et d'en comprendre l'ensemble en même temps que les détails. Mais, précisément, c'est là aussi ce qui lui donne un grand rendement. In *L'Avenir*, p. 931.

[10] J'ai parlé du sentiment de certitude absolue qui accompagne l'inspiration; dans les cas cités, ce sentiment n'était pas trompeur, et le plus souvent, il en est ainsi; mais il faut se garder de croire que ce soit une règle sans exception; souvent ce sentiment nous trompe sans pour cela être moins vif, et on ne s'en aperçoit que quand on cherche à mettre la démonstration sur pied. J'ai observé surtout le fait pour les idées qui me sont venues le matin ou le soir dans iroa lit, à l'état semi-hypnagogique. In Poincaré (1908c), rep. in Poincaré (1908a, pp. 55, 395). Page references of this kind refer to the French and English editions of the text where appropriate; I have used (Poincaré, 2001).