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First-order perturbative calculation of the frequency-shifts caused by static cylindrically-symmetric electric and magnetic imperfections of a Penning trap

Jochen Ketter*, Tommi Eronen, Martin Höcker, Sebastian Streubel, Klaus Blaum

Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany

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ABSTRACT

The ideal Penning trap consists of a uniform magnetic field and an electrostatic quadrupole potential. Cylindrically-symmetric deviations thereof are parametrized by the coefficients B_η and C_η, respectively. Relativistic mass-increase aside, the three characteristic eigenfrequencies of a charged particle stored in an ideal Penning trap are independent of the three motional amplitudes. This threefold harmonicity is a highly-coveted virtue for precision experiments that rely on the measurement of at least one eigenfrequency in order to determine fundamental properties of the stored particle, such as its mass. However, higher-order contributions to the ideal fields result in amplitude-dependent frequency-shifts. In turn, these frequency-shifts need to be understood for estimating systematic experimental errors, and eventually for correcting them by means of calibrating the imperfections. The problem of calculating the frequency-shifts caused by small imperfections of a near-ideal trap yields nicely to perturbation theory, producing analytic formulas that are easy to evaluate for the relevant parameters of an experiment. In particular, the frequency-shifts can be understood on physical rather than purely mathematical grounds by considering which terms actually drive them. Based on identifying these terms, we derive general formulas for the first-order frequency-shifts caused by any perturbation parameter B_η or C_η.

1. Introduction

Much more than a device for storing charged particles [1], the Penning trap excels at relating fundamental properties of the stored particle, such as its mass or magnetic moment, to a measurable frequency [2]. In order to make full use of the precision the Penning trap has to offer, the relationship between the measured frequency and the sought-for quantity has to be understood in detail despite the complications that come with a real-world experiment. Deviations from the ideal Penning trap may be unavoidable in general, but they can also serve a purpose as a part of the detection system [3,4].

In this paper, we employ a perturbative method to deal with one particularly important subset of imperfections—cylindrically-symmetric ones—and we focus on the frequency-shifts they cause. Although these are not the only consequence of imperfections, the frequency-shifts are often the most significant one, considering that frequencies constitute the main observables in a typical experiment.

Although the frequency-shifts for the experimentally most relevant lowest-order cylindrically-symmetric imperfections have previously been given numerous times [5–9], and the prescriptions for calculating all the first-order shifts caused by this subset of imperfections have been outlined in general [10–13], the specific formulas lack the common ground a general expression would provide. In this paper, we derive such readily-evaluated general expressions for all the first-order frequency-shifts caused by cylindrically-symmetric imperfections. As a little known fact, a general treatment of the problem has been attempted before [14] with Hamiltonian perturbation theory and classical canonical action-variables, but since we disagree with the result given for magnetic imperfections, a complete and correct check is certainly welcome. Moreover, we try to be more explicit about our calculation, thereby allowing the reader to verify its validity.

In Section 2, we review the most important properties of the ideal Penning trap as the zeroth-order input for the perturbative treatment of imperfections. Section 3 then deals with how to parametrize cylindrically-symmetric electric and magnetic imperfections. The mathematical groundwork for the calculation is laid in Section 4 with particular emphasis on the implementation of perturbation theory. With this method outlined in Section 4.2, the actual first-order frequency-shifts are subsequently calculated in

* Corresponding author. Tel.: +49 6221 516 683.
E-mail address: jochen.ketter@mpi-hd.mpg.de (J. Ketter).

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Sections 5 and 6 for electric and magnetic imperfections, respectively.

2. The ideal Penning trap

The ideal Penning trap consists of a homogeneous magnetic field $\vec{B}_0 = B_0\vec{e}_z$ pointing along the z-axis and an electrostatic quadrupole potential

$$\Phi_2(\rho, z) = \frac{V_0 C_2}{2d^2}\left(z^2 - \frac{\rho^2}{2}\right), \quad \text{where} \quad \rho = \sqrt{x^2 + y^2} \tag{1}$$

is the distance from the z-axis. In the context of the experiment, V_0 is understood as an applied voltage. The characteristic trap dimension d is typically defined such that the dimensionless parameter C_2 is close to unity for traps with hyperboloidal electrodes [15], but any value may be used to describe the quadrupole contribution in other trap geometries, such as cylindrical traps with flat-plate [16] or open endcaps [17].

Throughout this paper, we will work with the classical Newtonian equation of motion

$$\ddot{\vec{r}} = \frac{\vec{F}_\mathrm{L}}{m} = \frac{q}{m}(\vec{E} + \dot{\vec{r}} \times \vec{B}) \tag{2}$$

into which we insert the Lorentz force $\vec{F}_\mathrm{L}$ experienced by a point-like particle of mass m and charge q in the magnetic field $\vec{B}$ and the electric field $\vec{E} = -\vec{\nabla}\Phi$, derived by taking the negative gradient of the electrostatic potential Φ. Since the equation is linear in both fields, we will simply add imperfections as we go along.

While early treatment of the ideal Penning trap partly started out from a quantum-mechanical perspective [18,19], and an operator formalism suits the excitation and coupling of modes well [20,21], we will content ourselves with a purely classical model, ignoring both quantum-mechanical and relativistic effects. Spin and relativistic mass-increase can be treated as a perturbation of their own [5]. Furthermore, we will restrict ourselves to the "static" case, meaning that the particle oscillates with constant motional amplitudes in the absence of external excitation drives.

The emission of synchrotron radiation by an electron orbiting in a strong magnet field allows to cool the electron's cyclotron motion into its quantum-mechanical ground-state. For heavier particles, radiative cooling is inefficient [5], and the motional ground-state remains out of reach unless laser-cooling is used [22]. Typical other techniques such as buffer-gas cooling [23], resistive cooling of one motion via an LC tank circuit [24], and cooling via sideband-coupling to a cooled motion [25] leave the particle with high enough a set of quantum numbers to warrant a classical treatment. Moreover, some detection methods rely on motional amplitudes well above the thermal limit. It is only recently that quantum-jumps in the motion of a single resistively-cooled proton are on the brink of being resolved in a huge magnetic inhomogeneity, albeit as a spurious and ill-controlled side-effect where spin-flips are to be detected [26,27].

Throughout this paper, we will assume a charged particle devoid of internal degrees of freedom which could couple to electric or magnetic fields. Apart from spin, this also excludes polarizability, which may play the role of an effective mass [28]. For the ideal Penning trap with $\vec{B}_0 = B_0\vec{e}_z$ and $\vec{E}_2 = -\vec{\nabla}\Phi_2$, the classical equations of motion for a particle of charge q and mass m are

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \frac{qB_0}{m}\begin{pmatrix}\dot{y}\\ -\dot{x}\\ 0\end{pmatrix} + \frac{qV_0C_2}{2md^2}\begin{pmatrix}x\\ y\\ -2z\end{pmatrix}. \tag{3}$$

Being parallel to and therefore unaffected by the magnetic field, the axial motion is a one-dimensional harmonic oscillator with the angular frequency

$$\omega_z = \sqrt{\frac{qV_0C_2}{md^2}}. \tag{4}$$

Trapping requires $qV_0C_2 > 0$. If there was no electric field, the particle would orbit around the magnetic field-lines with the free-space cyclotron-frequency

$$\omega_\mathrm{c} = \frac{qB_0}{m}. \tag{5}$$

For $V_0 \neq 0$, the radial motion consists of two circular modes with frequencies[1]

$$\omega_\pm = \frac{1}{2}\left(\omega_\mathrm{c} \pm \frac{\omega_\mathrm{c}}{|\omega_\mathrm{c}|}\sqrt{\omega_\mathrm{c}^2 - 2\omega_z^2}\right) \tag{6}$$

Because the frequencies have to be real for the motion to stay bounded, the second condition for trapping is $|\omega_\mathrm{c}| > \sqrt{2}\omega_z$. The radial mode with the lower (absolute) frequency is called magnetron motion; the frequency ω_+ is associated with the modified cyclotron motion and also referred to as the reduced cyclotron-frequency because its absolute value is lower than the free-space cyclotron-frequency ω_c. In a typical experiment, the hierarchy is $|\omega_\mathrm{c}| \gtrsim |\omega_+| \gg \omega_z \gg |\omega_-|$.

The trajectory in the ideal Penning trap is given by

$$x(t) = \hat{\rho}_+ \cos(\omega_+ t + \varphi_+) + \hat{\rho}_- \cos(\omega_- t + \varphi_-), \tag{7}$$

$$y(t) = -\hat{\rho}_+ \sin(\omega_+ t + \varphi_+) - \hat{\rho}_- \sin(\omega_- t + \varphi_-), \tag{8}$$

$$z(t) = \hat{z}\cos(\omega_z t + \varphi_z). \tag{9}$$

The amplitudes $\hat{\rho}_\pm$ of the two radial modes and the amplitude $\hat{z}$ of the axial mode, as well as the corresponding initial phases φ_i with $i = (+, -, z)$ are determined by the initial conditions. Later on, we will use

$$\chi_i = \omega_i t + \varphi_i \tag{10}$$

as an abbreviation for the total phase without always stressing the time-dependent nature of χ_i.

From Eq. (6), we derive the three relations

$$\omega_+ + \omega_- = \omega_\mathrm{c}, \tag{11}$$

$$2\omega_+\omega_- = \omega_z^2, \tag{12}$$

$$\omega_+^2 + \omega_-^2 + \omega_z^2 = \omega_\mathrm{c}^2. \tag{13}$$

The first and the last identity are particularly important, not only because they relate the eigenfrequencies in the Penning trap to the

[1] In contrast to virtually all other publications, we have included essentially the sign of ω_c as a prefactor of the square root in Eq. (6), which allows us to handle negative cyclotron frequencies consistently. Whereas the sign of the angular frequency is not an additional degree of freedom for the one-dimensional axial motion and was consequently taken to be positive by convention, the sign of the angular frequencies associated with the two-dimensional radial motions encodes the sense of revolution in a natural manner. We do not have to think about the sign of the charge q or of the magnetic field B_0, which could point along the negative z-axis. Therefore, we will not work with true frequencies $\nu = |\omega|/(2\pi)$ in this paper. This is not meant to imply that the sense of rotation, defined in a coordinate system with either of two possible choices for the z-axis, impacts the frequency-shift—it does not, even less so for cylindrically-symmetric imperfections. However, the sign of the perturbation parameter B_η with respect to B_0 matters, and we do not want to run the risk of losing it while working with the absolute value of qB_0 in the free-space cyclotron-frequency ω_c. Moreover, the definition of $\omega_\pm$ with the additional factor $\omega_\mathrm{c}/|\omega_\mathrm{c}|$ ensures that $|\omega_+| \geq |\omega_-|$, regardless of the sign of ω_c.