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Classical calculation of relativistic frequency-shifts in an ideal Penning trap

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ABSTRACT

The ideal Penning trap consists of a uniform magnetic field and an electrostatic quadrupole potential. In the classical low-energy limit, the three characteristic eigenfrequencies of a charged particle trapped in this configuration do not depend on the amplitudes of the three eigenmotions. No matter how accurate the experimental realization of the ideal Penning trap, its harmonicity is ultimately compromised by special relativity. Using a classical formalism of first-order perturbation theory, we calculate the relativistic frequency-shifts associated with the motional degrees of freedom for a spinless particle stored in an ideal Penning trap, and we compare the results with the simple but surprisingly accurate model of relativistic mass-increase.

1. Introduction

Despite its versatility [1] and the eigenmotion called the modified cyclotron-mode, the Penning trap is not perceived as an accelerator—a device typically viewed as capable of producing highly energetic particles for which relativistic mass-increase plays an important role. Given the small scale of the Penning trap ranging from millimeters to a few centimeters, the charged particle stored in it may appear to move in the purely classical domain, well outside the realm of special relativity. In the classical limit, the three eigenfrequencies—all of which depend on the mass of the stored particle to a varying extent—are independent of the motional amplitudes. However, apart from possibly being too small to be detected, there is no threshold for the onset of relativistic effects and hence even the ideal Penning trap is inherently anharmonic. It is because of the outstanding precision of up to 10^{-10} for single frequency measurements that a relativistic shift was crucial to the determination of the antiproton mass [2]. Similarly, relativistic shifts may be dominant sources of uncertainty in measurements on light or highly charged ions [3–5]. Conversely, these shifts are particularly interesting for measuring motional amplitudes [6,7] because, unlike the anharmonic shifts caused by other imperfections, they do not depend on specific parameters of the trap apart from the readily measured frequencies.

Probably because of early work on electrons and the interest in their magnetic moment [8], the theoretical treatment of relativistic frequency-shifts used quantum-mechanical operator formalisms [9–12]. When relativistic equations of motion were considered [13–15], the focus was more on excitations of the modified cyclotron-mode than on static frequency-shifts.

A classical treatment with relativistic additions does not have to be conceptually inferior to a fully relativistic or quantum-mechanical approach, in particular if quantization remains unobservable and an exact solution is impossible in either case. In fact, reproducing the classical limit is in general a benchmark for a relativistic quantum theory. Consequently, knowing the prediction of a non-quantized treatment is worthwhile.

In this paper, we show that the relativistic frequency-shifts caused by the motional degrees of freedom of a charged particle stored in an ideal Penning trap are also reproduced in a classical framework of perturbation theory. With classical framework we refer to the use of equations of motion in contrast to operators and eigenstates. In Section 2, we approximate the relativistic equation of motion such that classical perturbation theory can be applied with the classical limit of the ideal Penning trap as the starting point. We also outline our particular implementation of first-order perturbation theory. The actual relativistic frequency-shifts are calculated in Section 3. In Section 4, the result is then compared with a simple model of relativistic mass-increase.

2. Theory and method

The theoretical framework of perturbation theory is essentially the same as the one we used to calculate the first-order frequency-shifts caused by static cylindrically symmetric electric

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and magnetic imperfections of a Penning trap [16]. This time, we have to learn how to incorporate relativistic effects as a perturbation in the classical equations of motion. To this end, we take a more general look at the relativistic equations of motion in search of a suitable perturbation parameter, before plugging in the specific electric and magnetic field of the ideal Penning trap.

2.1. Relativistic equation of motion

Consider a static electric field $\vec{E}$ and a static magnetic field $\vec{B}$ in the laboratory frame. We will use this frame exclusively throughout the paper, never looking at the particle's rest frame or its proper time. Accordingly, all time-derivatives shown are with respect to time in the laboratory frame. Moreover, the limitation to static fields spares us from the complications of retardation. We will also ignore radiation damping because the emission of synchrotron radiation is insignificant for particles heavier than electrons [11]. For a pointlike spinless particle of charge q and rest mass m, the relativistic equation of motion is then given by

$$\dot{\vec{p}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\vec{v}) = q(\vec{E} + \vec{v}\times\vec{B}), \tag{1}$$

where $\vec{p}$ is the particle's momentum and $\vec{v}$ its velocity. We will use $p = |\vec{p}|$ and $v = |\vec{v}|$ as an abbreviation for the length of the corresponding vectors. Thus far, the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ with the speed of light c is the only difference from the classical Newtonian equation of motion. However, in addition to the familiar acceleration $\dot{\vec{v}}$, taking the time-derivative of the relativistic momentum $\vec{p} = \gamma m\vec{v}$ results in a time-derivative of the Lorentz factor, which is expressed more conveniently via the particle's total energy $\mathcal{E} = \gamma mc^2$ and the relativistic energy–momentum relation as

$$\dot{\gamma} = \frac{1}{mc^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \frac{1}{mc^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{(mc^2)^2 + (pc)^2}\right] = \frac{\dot{\vec{p}}\cdot\vec{p}}{\gamma m^2 c^2}. \tag{2}$$

By plugging in the right-hand side of Eq. (1) for $\dot{\vec{p}}$ and by recalling that the momentum $\vec{p}$ is always perpendicular to the force $q\vec{v}\times\vec{B}$ associated with the magnetic field, the relativistic equation of motion is rewritten as

$$\dot{\vec{v}} = \frac{q}{\gamma m}\left(\vec{E} + \vec{v}\times\vec{B}\right) - \frac{q}{\gamma mc^2}\vec{v}(\vec{E}\cdot\vec{v}). \tag{3}$$

Apart from the Lorentz factor γ, which might be understood as relativistic mass-increase by redefining the mass as $m \to \gamma m$, there is an additional term that is not present in the classical Newtonian equations of motion. However, these are recovered in the classical limit of $c \to \infty$, and consequently $\gamma \to 1$.

The ideal Penning trap consists of a homogeneous magnetic field $\vec{B}_0 = B_0\vec{e}_z$ that is perfectly aligned along the z-axis and an electrostatic field

$$\vec{E}_2 = -\vec{\nabla}\Phi_2 = \frac{V_0}{2d^2}\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} \tag{4}$$

that is derived from the quadrupole potential

$$\Phi_2 = \frac{V_0}{2d^2}\left(z^2 - \frac{x^2+y^2}{2}\right). \tag{5}$$

The voltage V_0 and the characteristic dimension d determine the strength of the electric field gradient.

Whereas we will present an analytic solution for the classical equations of motion in the ideal Penning trap shortly, no such general solution is possible for the fully relativistic case because of the coupling introduced by the Lorentz factor γ. The situation is identical to the quantum-mechanical case: the Schrödinger Hamiltonian is treated analytically in terms of three uncoupled harmonic oscillators [17,18], but no exact solution for the relativistic wave equations of a charged particle in a Penning trap is known. Either way, approximations have to be made when relativistic effects are taken into account. Since the motion of a charged particle stored in a Penning trap is typically only barely relativistic, a perturbative treatment of the lowest-order relativistic corrections suffices.

With this simplification in mind, we adapt the relativistic equation of motion (3) accordingly, by expanding the inverse of the Lorentz factor γ as

$$\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{v^2}{2c^2} - \cdots \tag{6}$$

for small velocities $v \ll c$, thereby effectively assigning the role of a perturbation parameter to c^{-2}. By ignoring all the terms of higher order than c^{-2}, such as the next order c^{-4}, the equation of motion reads

$$\dot{\vec{v}} = \frac{q}{m}\left(1 - \frac{v^2}{2c^2}\right)\left(\vec{E} + \vec{v}\times\vec{B}\right) - \frac{q}{mc^2}\vec{v}(\vec{E}\cdot\vec{v}). \tag{7}$$

Note that—as an intrinsically relativistic correction—the last term in Eq. (3) already came with a factor c^{-2}. Therefore, already the lowest-order relativistic correction in the Lorentz factor γ results in a term of order c^{-4}, which we have neglected here.

By inserting the electric field $\vec{E}_2$ given in Eq. (4) and the uniform magnetic field $\vec{B}_0 = B_0\vec{e}_z$ of the ideal Penning trap into Eq. (7), the approximate equations of motion become

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \left[1 - \frac{v^2}{2c^2}\right]\frac{\omega_z^2}{2}\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} + \left[1 - \frac{v^2}{2c^2}\right]\omega_c\begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix} - \frac{1}{c^2}\frac{\omega_z^2}{2}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}[\dot{x}x + \dot{y}y - 2\dot{z}z] \tag{8}$$

with the velocity squared

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{9}$$

given by the quadratic sum of the individual components of the velocity vector $\vec{v}$. As abbreviations related to the classical case, we have introduced the free-space cyclotron-frequency

$$\omega_c = \frac{qB_0}{m} \tag{10}$$

with which the particle would orbit around the magnetic field-lines if there were no electric field, and the axial frequency

$$\omega_z = \sqrt{\frac{qV_0}{md^2}} \tag{11}$$

with which the particle oscillates along the magnetic field-lines. Clearly, trapping requires $qV_0 > 0$.

In the classical limit of $c \to \infty$, the radial motion of the particle consists of two circular modes with frequencies

$$\omega_\pm = \frac{1}{2}\left(\omega_c \pm \frac{\omega_c}{|\omega_c|}\sqrt{\omega_c^2 - 2\omega_z^2}\right), \tag{12}$$

where ω_+ is called the reduced or modified cyclotron-frequency, and ω_- represents the magnetron frequency. Trapping requires $\omega_c^2 > 2\omega_z^2$. The frequencies in the ideal classical trap satisfy the relations

$$2\omega_+\omega_- = \omega_z^2, \tag{13}$$