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# Extensions of a result by G. Talenti to (p,q)-Laplace equations

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ABSTRACT

We prove a comparison result for solutions to (p,q)-Laplace equation via Schwarz symmetrization. For the p-Laplace equation, the corresponding result is due to Giorgio Talenti. In a special (radial) case we also prove a reverse comparison result.


## 1. Introduction

An inequality attributed to Giorgio Talenti [11] has been a masterpiece in partial differential equation. Its physical and mathematical implications are profound. The original description of this inequality is here reviewed for the convenience of the reader. Let $u \in H_0^1(\Omega)$ be the solution to the boundary value problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + c(x)u = f(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $a_{ij}(x),\ c(x)$ are measurable functions satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \sum_{i=1}^{n} \xi_i^2, \quad c(x) \geq 0,$$

and $f \in L_+^\infty(\Omega)$. Henceforth, $\Omega \subset \mathbb{R}^n$ is a bounded domain.

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On the other hand, let $v \in H_0^1(B)$ be the solution to the following Poisson boundary value problem:

$$-\Delta v = f^\sharp(x) \ \text{ in } \ B, \quad v = 0 \ \text{ on } \ \partial B, \tag{2}$$

where $B$ is the ball in $\mathbb{R}^n$ centered at the origin and such that $|B| = |\Omega|$ (the radius $R$ of $B$ is $(|\Omega|/\omega_n)^{1/n}$, where $\omega_n$ stands for the measure of the n-dimensional unit ball). The notation $f^\sharp$ stands for the Schwarz symmetrization of $f$. In other words, $f^\sharp(x) = f^*(\omega_n|x|^n)$, where $f^*$ denotes the essentially unique decreasing rearrangement of $f$ defined on the interval $[0, |\Omega|]$. The Talenti inequality follows:

$$u^\sharp(x) \le v(x) \ \text{ for almost every } \ x \in B. \tag{3}$$

Some immediate consequences of (3) are summarized below.
(i) ess $\sup_\Omega u(x) \le$ ess $\sup_B v(x)$,
(ii) $\int_\Omega u^r\, dx \le \int_B v^r\, dx, \ 1 \le r < \infty$ (norm estimate),
(iii) $\int_\Omega \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j}\, dx \le \int_B |\nabla v|^2\, dx$ (energy estimate).

The most common version of Talenti's inequality corresponds to the case when $a_{ij}(x) = \delta_{ij}$, and the potential function $c(x)$ is identically zero. Under these conditions, problem (1) reduces to a Poisson problem:

$$-\Delta u = f(x) \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{4}$$

In this case, when $n = 2$, (3) can be interpreted physically as follows. An elastic membrane occupying the (horizontal) region $\Omega$, fixed at the boundary $\partial\Omega$, and subject to a vertical force $f(x)$ is displaced from the rest position. The amount of displacement is denoted by $u$, the solution to (4). In this setting, the Talenti inequality (3), guarantees that a circular membrane having the same area as that of $\Omega$, and subject to a radial vertical force but equi-measurable with $f(x)$ will have the largest displacement. Applications of Talenti's inequality in the literature are overwhelming, the reader is suggested to refer to a recent survey [12], and the numerous references therein for a thorough treatment.

In the present paper, we prove an extension of Talenti's inequality (3). The precise description of the problem of interest is as follows. Consider the boundary value problem

$$-\Delta_p u - \Delta_q u = f(x)h(u), \quad u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \tag{5}$$

where $1 < q < p$, $f(x)$ is a non-negative bounded function, and $h : (0, \infty) \to (0, \infty)$ satisfies the conditions:
(H1) $h(t)$ is a non-decreasing function,
(H2) $h(t)t^{1-\alpha}$ is non-increasing for some $\alpha$ such that $1 \le \alpha < q$.

The operator $\Delta_p$ (similarly $\Delta_q$) denotes the usual p-Laplacian i.e. $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$. Our first result is the following.

**Theorem 1.1.** *Let $u$ be the unique positive solution to (5). Suppose $v$ is the solution to*

$$-\Delta_p v - \Delta_q v = f^\sharp(x)h(v), \quad v > 0 \ \text{ in } \ B, \quad v = 0 \ \text{ on } \ \partial B, \tag{6}$$

*where $B$ is the ball centered at the origin with radius $R = (|\Omega|/\omega_n)^{1/n}$. Then*

$$u^\sharp(x) \le v(x) \ \text{ for almost every } \ x \in B.$$

The proof of the above theorem relies on a comparison result (see Proposition 2.1) which is itself of interest. Indeed, it is this comparison result which guarantees the solution to problem (5) to be unique, whereas the existence of a positive solution follows from standard variational arguments (note that, thanks to condition (H2), the associated energy integral is coercive).