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Existence of waves for a reaction–diffusion–dispersion system

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A B S T R A C T

Existence of travelling waves is studied for a bistable reaction–diffusion system of equations with linear integral terms (dispersion) and with some conditions on the nonlinearity. The proof is based on the Leray–Schauder method using the topological degree theory for Fredholm and proper operators with the zero index and a priori estimates of solutions in properly chosen weighted spaces.

1. Introduction

Conventional diffusion terms in reaction–diffusion equations describe random motion of atoms and molecules in multi-component continuous media or random motion of individuals in population dynamics. There exist also other types of motion in various applications, including a long range dispersion in ecology [1,2], in neural models [3,4] or in phase field models [5]. Different mechanisms of motion can be alternated or combined. For example, long range dispersion for animal nesting is accompanied by a short-range random motion for foraging.

In this work we present a method to study the existence of travelling waves of reaction–diffusion–dispersion equations. Consider the system of equations

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + SJ(u) + F(u), \tag{1.1}$$

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where $u = (u_1, \ldots, u_n)$, $F = (F_1, \ldots, F_n)$, $J = (J_1, \ldots, J_n)$,

$$J_k(u) = \int_{-\infty}^{\infty} \phi_k(x-y) u_k(y,t) dy \ , \quad k = 1, \ldots, n,$$

$\phi_k(x)$ are non-negative even functions such that $\phi_k(x)\exp(r|x|)$ are integrable for some $r > 0$, $\int_{-\infty}^{\infty} \phi_k(x)dx = 1$; D is a diagonal matrix with positive diagonal elements d_i, S is a matrix with constant elements $s_{ij} \geq 0$, $i, j = 1, \ldots, n$.

We will consider in this work system (1.1) on the whole axis, $x \in \mathbb{R}$, and will study the existence of travelling wave solutions. It is a solution $u(x,t) = w(x - ct)$, which satisfies the second-order system of equations

$$Dw'' + SJ(w) + cw' + F(w) = 0, \tag{1.2}$$

where c is an unknown constant, the wave speed. We will look for solutions with some limits at infinity,

$$w(\pm\infty) = w_{\pm}, \tag{1.3}$$

where $w_{\pm}$ are solutions of the equation

$$\Phi(w) \equiv Sw + F(w) = 0. \tag{1.4}$$

We assume that $0 < w_+ < w_-$ (the inequalities are understood component-wise), and we will consider monotonically decreasing solutions of problem (1.2), (1.3). Next, suppose that the vector-function $F(w)$ is continuous together with its second derivatives and satisfies the following condition:

$$F_i(w) \leq 0 \quad \Rightarrow \quad \frac{\partial F_i(w)}{\partial u_j} > 0 \ , \quad j \neq i, \ j = 1, \ldots, n. \tag{1.5}$$

This condition is automatically satisfied for the scalar equation ($n = 1$). Let us recall the definitions of some related classes of systems.

Monotone systems. The system is called monotone if the right-hand side inequality in (1.5) is satisfied for all $w \in R^n$. This is the class of systems for which the maximum principle is applicable. Existence, uniqueness and stability of waves for the monotone reaction–diffusion systems without dispersion ($S = 0$) was studied in [6,7]. The scalar equation with dispersion was considered in [5,8]. There are numerous works with a nonlinear dependence on the integral terms (see [9] and the references therein). There are various applications of such systems in chemical kinetics and combustion, in population dynamics and biomedicine [7,10].

Locally monotone systems. The reaction–diffusion system (without dispersion) is called locally monotone if the inequality $\frac{\partial F_i(w)}{\partial u_j} > 0$ holds for $j \neq i$, $j = 1, \ldots, n$ and for such w that $F_i(w) = 0$ [6,7] (p. 154). Thus, the inequality should be satisfied only on zero surfaces of the functions $F_i(w)$ and not everywhere. It is a more general class of systems (than monotone systems) also encountered in various applications. Let us note that if a function $F(w)$ satisfies the monotonicity condition, then the function $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_n)$, $\tilde{F}_i = g_i(w)F_i(w)$ is locally monotone for any $g_i(w) > 0$. This remark opens a wide range of applications of such systems. Locally monotone systems do not satisfy the maximum principle. The existence of waves for such systems is proved in [6,7,11], the stability of waves, in general, may not hold. Locally monotone systems with dispersion are not studied since the same methods are not applicable for them.

In this work we study locally monotone systems with dispersion. Local monotonicity is understood here in the sense of condition (1.5). The presence of the integral terms makes this condition more restrictive in comparison with the reaction–diffusion systems without dispersion. The proof of the wave existence is based on the Leray–Schauder method using the topological degree for elliptic operators in unbounded domains and a priori estimates of solutions in some weighted spaces. The degree is constructed for Fredholm and proper operators with the zero index [6,12] (Section 2.2). The main result of this work is given in the following theorem.