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# On the nonexistence of global solutions for critical semilinear wave equations with damping in the scattering case

Kyouhei Wakasa [a,*], Borislav Yordanov [b,c]

[a] Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda-shi, Chiba, 278-8510, Japan
[b] Office of International Affairs, Hokkaido University, Kita 15, Nishi 8, Kita-ku, Sapporo, Hokkaido 060-0815, Japan
[c] Institute of Mathematics, Sofia, Bulgaria

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ABSTRACT

We consider the Cauchy problem for semilinear wave equations with variable coefficients and time-dependent scattering damping in $\mathbf{R}^n$, where $n \geq 2$. It is expected that the critical exponent will be Strauss' number $p_0(n)$, which is also the one for semilinear wave equations without damping terms.

Lai and Takamura (2018) have obtained the blow-up part, together with the upper bound of lifespan, in the sub-critical case $p < p_0(n)$. In this paper, we extend their results to the critical case $p = p_0(n)$. The proof is based on Wakasa and Yordanov (0000), which concerns the blow-up and upper bound of lifespan for critical semilinear wave equations with variable coefficients.


## 1. Introduction

We study the blow-up problem for critical semilinear wave equations with variable coefficients and scattering damping depending on time. The perturbations of Laplacian are uniformly elliptic operators

$$\Delta_g = \sum_{i,j=1}^{n} \partial_{x_i} g_{ij}(x) \partial_{x_j}$$

whose coefficients satisfy, with some $\alpha > 0$, the following:

$$g_{ij} \in C^1(\mathbf{R}^n), \quad |\nabla g_{ij}(x)| + |g_{ij}(x) - \delta_{ij}| = O(e^{-\alpha|x|}) \text{ as } |x| \to \infty. \tag{1.1}$$

The admissible damping coefficients are $a \in C([0,\infty))$, such that

$$\forall t \geq 0 \quad a(t) \geq 0 \ \text{ and } \int_0^\infty a(t)dt < \infty. \tag{1.2}$$

* Corresponding author.
*E-mail addresses:* wakasa_kyouhei@ma.noda.tus.ac.jp (K. Wakasa), byordanov@oia.hokudai.ac.jp (B. Yordanov).

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For $n \geq 2$ and $p > 1$, we consider the Cauchy problem

$$u_{tt} - \Delta_g u + a(t)u_t = |u|^p, \; x \in \mathbf{R}^n, \quad t > 0, \tag{1.3}$$

$$u|_{t=0} = \varepsilon u_0, \quad u_t|_{t=0} = \varepsilon u_1, \; x \in \mathbf{R}^n, \tag{1.4}$$

where $u_0$, $u_1 \in C_0^\infty(\mathbf{R}^n)$ and $\varepsilon > 0$ is a small parameter. Our results concern only the critical case $p = p_0(n)$ with Strauss' exponent defined in (1.5).

Let us briefly review previous results concerning (1.3) with $g_{ij} = \delta_{ij}$ and various types of damping $a$. When $a(t) = 1$, Todorova and Yordanov [13] showed that the solution of (1.3) blows up in finite time if $1 < p < p_F(n)$, where $p_F(n) = 1 + 2/n$ is the Fujita exponent known to be the critical exponent for the semilinear heat equation. The same work also obtained global existence for $p > p_F(n)$. Finally, Zhang [20] established the blow-up in the critical case $p = p_F(n)$.

The other typical example of effective damping is $a(t) = \mu/(1+t)^\beta$ with $\mu > 0$ and $\beta \in \mathbf{R}$. When $-1 < \beta < 1$, Lin, Nishihara and Zhai [9] obtained the expected blow-up result, if $1 < p \leq p_F(n)$, and global existence result, if $p > p_F(n)$; see also D'Abbicco, Lucente and Reissig [2].

In the case of critical decay $\beta = 1$, there are several works about finite time blow-up and global existence. Wakasugi [17] showed the blow-up, if $1 < p \leq p_F(n)$ and $\mu > 1$ or $1 < p \leq 1 + 2/(n + \mu - 1)$ and $0 < \mu \leq 1$. Moreover, D'Abbicco [1] verified the global existence, if $p > p_F(n)$ and $\mu$ satisfies one of the following: $\mu \geq 5/3$ for $n = 1$, $\mu \geq 3$ for $n = 2$ and $\mu \geq n + 2$ for $n \geq 3$. An interesting observation is that the Liouville substitution $w(x,t) := (1+t)^{\mu/2}u(x,t)$ transforms the damped wave equation (1.3) into the Klein–Gordon equation

$$w_{tt} - \Delta w + \frac{\mu(2-\mu)}{4(1+t)^2}w = \frac{|w|^p}{(1+t)^{\mu(p-1)/2}}.$$

Thus, one expects that the critical exponent for $\mu = 2$ is related to that of the semilinear wave equation. D'Abbicco, Lucente and Reissig [3] have actually obtained the corresponding blow-up result, if $1 < p < p_c(n) := \max\{p_F(n),\; p_0(n+2)\}$ and

$$p_0(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n-1)} \tag{1.5}$$

is the so-called Strauss exponent, the positive root of the quadratic equation

$$\gamma(p,n) := 2 + (n+1)p - (n-1)p^2 = 0. \tag{1.6}$$

Their work also showed the existence of global classical solutions for small $\varepsilon > 0$, if $p > p_c(n)$ and either $n = 2$ or $n = 3$ and the data are radially symmetric. Finally, we mention that our original equations (1.3) are related to semilinear wave equations in the Einstein–de Sitter spacetime considered by Galstian and Yagdjian [4].

We recall that $p_0(n)$ in (1.5) is the critical exponent for the semilinear wave equation conjectured by Strauss [11]. The hypothesis has been verified in several cases; see [16] and the references therein. A related problem is to estimate the lifespan, or the maximal existence time $T_\varepsilon$ of solutions to (1.3), (1.4) in the energy space $C([0,T_\varepsilon), H^1(\mathbf{R}^n)) \cap C^1([0,T_\varepsilon), L^2(\mathbf{R}^n))$.

Lai, Takamura and Wakasa [8] have obtained the blow-up part of Strauss' conjecture, together with an upper bound of the lifespan $T_\varepsilon$, for (1.3), (1.4) in the case $n \geq 2$, $0 < \mu < (n^2 + n + 2)/2(n+2)$ and $p_F(n) \leq p < p_0(n + 2\mu)$. Later, Ikeda and Sobajima [5] were able to replace these conditions by less restrictive $0 < \mu < (n^2 + n + 2)/(n+2)$ and $p_F(n) \leq p \leq p_0(n + \mu)$. In addition, they have derived an upper bound on the lifespan. Tu and Lin [14,15] have improved the estimates of $T_\varepsilon$ in [5] recently.

For $\beta \leq -1$, the long time behavior of solutions to (1.3), (1.4) is quite different. When $\beta = -1$, Wakasugi [18] has obtained the global existence for exponents $p_F(n) < p < n/[n-2]_+$, where

$$[n-2]_+ := \begin{cases} \infty & \text{for} \quad n = 1, 2, \\ n/(n-2) & \text{for} \quad n \geq 3. \end{cases}$$

Ikeda and Wakasugi [6] have proved that the global existence actually holds for any $p > 1$ when $\beta < -1$.