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Representation of lattices via set-colored posets

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ABSTRACT

This paper proposes a representation theory for any finite lattice via set-colored posets, in the spirit of Birkhoff for distributive lattices. The notion of colored posets was introduced in Nourine (2000) [34] and the generalization to set-colored posets was given in Nourine (2000) [35]. In this paper, we give a characterization of set-colored posets for general lattices, and show that set-colored posets capture the order induced by join-irreducible elements of a lattice as Birkhoff's representation does for distributive lattices. We also give a classification for some lattices according to the coloring property of their set-colored representation including upper locally distributive, upper locally distributive, meet-extremal and semidistributive lattices.

1. Introduction

All the objects considered are finite. This paper is motivated by representation theory and its algorithmic consequences for combinatorial objects structured as finite lattices. Whenever you are familiar with Birkhoff's theorem, the intuition behind this new representation is the following: *Take a poset, say $P = (X, \leq)$, a set of colors M and color the elements of P by subsets of M. Then the set of all ideal colors sets of P has a lattice structure and every lattice can be obtained in this way. For example if each element of P has exactly one color then the obtained family of ideal colors sets corresponds to an antimatroid. Moreover, if any two elements have different colors, then the lattice is distributive.*

The question "*Why develop lattice theory?*" was considered by Birkhoff [3,5] and extended by Wille [39,14] using Formal Concept Analysis (FCA).

The purpose of this paper is to present a representation of lattices that allows us to understand lattice theory. Let us first recall the famous Birkhoff's representation theorem for distributive lattices. All results in this paper can be stated for the dual, by replacing join-irreducible with meet-irreducible elements.

Theorem 1 ([4]). *Any distributive lattice L is isomorphic to the lattice of all order ideals $\mathcal{I}(J(L))$, where $J(L)$ is the poset induced by the set of join-irreducible elements of L.*

Birkhoff's theorem has been widely used to derive algorithms in many areas. In fact, whenever a set of objects has a structure isomorphic to a distributive lattice, then there exists a poset where the set of its order ideals is isomorphic to the lattice, e.g. stable marriages [18], stable allocation [2], minimum cuts in a network [22], etc.

For general lattices, the unique well known representation is based on sets [6] or a binary relation between join-irreducible and meet-irreducible elements, called the bipartite irreducible poset $Bip(L) = (J(L), M(L), \not\leq_L)$ by Markowsky

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J/M	1	2	3	4	5
a	1	1	0	1	1
b	1	0	0	1	1
c	0	1	1	1	1
d	0	1	1	1	0
e	1	1	1	0	1
f	1	0	1	0	1

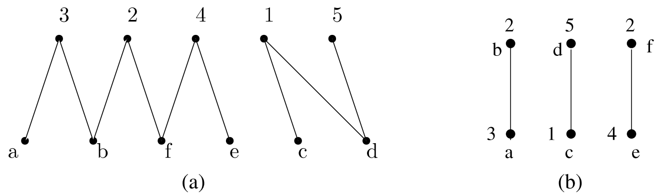

Fig. 1. The table shows a context or binary relation: (a) its corresponding bipartite irreducible poset, and (b) its corresponding set-colored poset.

[30–32] or a context $B(L) = (J(L), M(L), \leq_L)$ by Wille in the framework of Formal Concept Analysis (FCA) [14,39]. From the FCA perspective, elements in $J(L)$ are interpreted as objects and those of $M(L)$ are understood as properties or attributes characterizing these objects. Several other lattice representations have been proposed such as closure systems, implicational systems, and join-core, but all of them use exponential size (see [11,9,24,28]). Lattices representation, FCA and its algorithmic aspects are essential topics in data analysis, as they aim at identifying knowledges and restructuring them as a hierarchy (the reader is referred for examples to the ICFCA series of conferences). Over the past two decades, many algorithms have been introduced to consider the reconstruction of a lattice from its representation. Fortunately there exist linear time reconstruction algorithms for distributive lattice (see for example [19,23,38]). Enumeration or reconstruction algorithms for general lattices are equivalent to enumerate maximal bicliques of a bipartite graph (see [15] for a detailed analysis).

Compared to the representation of distributive lattices, the bipartite irreducible poset or the context does not take into account the order inherited by join-irreducible or meet-irreducible elements and the fact that the elements of the lattice correspond to some order ideals of the poset $J(L) = (J(L), \leq)$. In this sense the proposed algorithms for the general case have a bad behavior whenever the lattice is distributive or near to be distributive (e.g. upper locally distributive, semidistributive, …).

In this paper, we derive a new representation for general lattices via set-colored posets which generalizes the notion of colored posets for representing upper locally distributive lattice in [34,35]. For a lattice L, this representation captures the order inherited by join-irreducible elements $J(L)$ as Birkhoff's representation does for distributive lattices. When restricted to distributive lattices, we obtain an isomorphism between L and the order ideals of $J(L)$. Dilworth [10] has introduced upper locally distributive lattices and has observed that they are close to distributive lattices. Using set-colored posets we confirm this idea and obtain a characterization strongly linked to that of distributive lattices, which can be also deduced from Korte and Lovász [27] and Edelman and Sacks [12] works. Recently, Knauer [26,25] has confirmed this observation using antichains partition. Moreover Magnien et al. [29] have shown that configurations of a Chip-Firing game are structured as upper locally distributive lattice (see Kolja [26]).

Fig. 1 shows an example of a set-colored poset corresponding to an antimatroid. The interest of this representation is to consider only the poset induced by join-irreducible elements and to color them with some particular meet-irreducible elements. Indeed, the colors of a join-irreducible j is all meet-irreducible m such that $j \swarrow m$ (see later). Clearly this representation uses less space memory.

The remainder of the paper is structured as follows: in Section 2, we introduce the set-colored posets and the lattice of ideal colors sets. Section 3 gives a characterization of set-colored posets which are associated to a lattice. In Section 4, we discuss characterization of particular lattices that are close to distributive lattices.

2. Set-colored posets

In this section we first introduce the notion of set-colored posets and some notations that will be used throughout this paper. For definitions on lattices and ordered sets not given here, see [9,14,37].

A partial order (or poset) on a set X is a binary relation $\leq$ on X which is reflexive, anti-symmetric and transitive, denoted by $P = (X, \leq)$. A set $I \subseteq X$ is said to be an ideal if $x \in I$ and $y \leq x$ implies $y \in I$. For an element $x \in X$ we associate the unique ideal $\downarrow x = \{y \in X \mid y \leq x\}$. The set of all ideals of P is denoted by $\mathcal{I}(P)$.

Let $A \subseteq X$, an element $z \in X$ is an upper bound of A if $x \leq z$ for any $x \in A$. The element z is the least upper bound if $z \leq z'$ for all upper bounds z' of A. Dually, we define the greatest lower bound. A partial order $L = (X, \leq)$ is called a lattice if for every two elements $x, y \in X$ both the least upper bound and the greatest lower bound exist, denoted by $x \vee y$ and $x \wedge y$.

Let $L = (X, \leq)$ be a lattice. The element $z \in X$ is a join-irreducible (resp. meet-irreducible) if $z = x \vee y$ (resp. $z = x \wedge y$) implies $z = x$ or $z = y$. The set of all join-irreducible (resp. meet-irreducible) elements of L is denoted by $J(L)$ (resp. $M(L)$).

Let L be a finite lattice and $x, y \in L$. We will use the following arrow relations [14], that are weakening of the so called perspectivity relations defined in lattices (see [17]): $x \swarrow y$ means that x is a minimal element of $\{z \in L \mid z \not\leq y\}$, $x \searrow y$