Contents lists available at ScienceDirect
Discrete Applied Mathematics
journal homepage: www.elsevier.com/locate/dam

A single axiom for Boolean algebras

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ARTICLE INFO

Article history:
Received 15 February 2016
Received in revised form 28 December 2017
Accepted 7 August 2018
Available online xxxx

Keywords:
Lattices
Boolean algebras
Weak complementations
Concept algebras

ABSTRACT

We prove that an algebra $(L, \wedge, \vee, ')$ of type $(2, 2, 1)$ is a Boolean algebra iff $(L, \wedge, \vee)$ is a non empty lattice that satisfies the equation

$$(x \wedge y) \vee (x \wedge y') = (x \vee y) \wedge (x \vee y'). \qquad (\ddagger)$$

1. Introduction

Boolean algebras are well-known and widely investigated structures with applications in several domains: mathematics, computer science, statistics, ... They arise from the investigation of the laws of thought by George Boole [1] and can be defined by the set of equations in Definition 1. *Boolean algebras* have two binary operations $\vee$ and $\wedge$ encoding disjunction and conjunction respectively, and a unary operation $'$ encoding negation. To verify whether a given algebra is a Boolean algebra, we need not have to check all the ten axioms (and their duals) in Definition 1. For example it is enough to check the six Huntington's postulates [3]. By restricting Definition 1 to the first four equations we get a larger class of algebras called *lattices*. The result we are going to present in this paper characterizes Boolean algebras inside this class using a single equation. There are some similar results characterizing Boolean algebras inside the de Morgan algebras and orthomodular lattices [10], and inside bounded lattices [4]. Our result was obtained independently from these, as we were investigating the class of weakly dicomplemented lattices, introduced to capture the equational theory of *concept algebras* (see Section 4). This paper is a revised version of [5]. We will try, as much as possible, to make the paper self contained. Section 2 is devoted to weakly dicomplemented lattices and their connection to Boolean algebras. The main contribution is Theorem 3. Section 3 briefly discusses the connection with other works. Section 4 introduces the concept algebras, and by then give a wide class of examples of weakly dicomplemented lattices. We finish Section 1 with some basic definitions.

Definition 1. A **Boolean algebra** is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following equations:

(1) $x \vee x = x$,	(1') $x \wedge x = x$
(2) $x \vee y = y \vee x$	(2') $x \wedge y = y \wedge x$
(3) $x \vee (y \vee z) = (x \vee y) \vee z$,	(3') $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(4) $x \vee (x \wedge y) = x$,	(4') $x \wedge (x \vee y) = x$.
(5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,	(5') $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

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https://doi.org/10.1016/j.dam.2018.08.002

Please cite this article in press as: L. Kwuida, A single axiom for Boolean algebras, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.08.002.

(6) $x \vee 1 = 1$, (6') $x \wedge 0 = 0$
(7) $x \vee 0 = x$, (7') $x \wedge 1 = x$
(8) $x \vee x' = 1$, (8') $x \wedge x' = 0$,
(9) $x'' = x$,
(10) $(x \vee y)' = x' \wedge y'$ (10') $(x \wedge y)' = x' \vee y'$.

A **lattice** is an algebra $(L, \vee, \wedge)$ satisfying Eqs. (1), (2), (3), (4) and their duals. Eq. (5) and its dual are the **distributive laws**. A lattice is **bounded** if it satisfies the equations (6), (7) and their duals. A lattice is **complemented** if it satisfies equations (8) and (8'). In this case x' is called a **complement** of x. The complement needs not be unique if the lattice is not distributive. Eq. (10) and (10') are called the **De Morgan's laws**. Note that an algebra $(L, \vee, \wedge, ', 0, 1)$ is a Boolean algebra iff it satisfies (2), (5), (7), (8) and their duals (**Huntington's postulates** [3]).

2. Weakly dicomplemented lattices

Weakly dicomplemented lattices were introduced to capture the equational theory of concept algebras.

Definition 2. A **weakly dicomplemented lattice** is a bounded lattice L equipped with two unary operations $^\triangle$ and $^\triangledown$ called **weak complementation** and **dual weak complementation**, and satisfying for all $x, y \in L$ the conditions:

(11) $x^{\triangle\triangle} \leq x$, (11') $x^{\triangledown\triangledown} \geq x$,
(12) $x \leq y \Longrightarrow x^\triangle \geq y^\triangle$, (12') $x \leq y \Longrightarrow x^\triangledown \geq y^\triangledown$,
(13) $(x \wedge y) \vee (x \wedge y^\triangle) = x$, (13') $(x \vee y) \wedge (x \vee y^\triangledown) = x$.

We call $x^\triangle$ the **weak complement** of x and $x^\triangledown$ the **dual weak complement** of x. The pair $(x^\triangle, x^\triangledown)$ is called the **weak dicomplement** of x and the pair $(^\triangle, ^\triangledown)$ a **weak dicomplementation** on L. The structure $(L, \wedge, \vee, ^\triangle, 0, 1)$ is called a **weakly complemented lattice** and $(L, \wedge, \vee, ^\triangledown, 0, 1)$ a **dual weakly complemented lattice**.

Note that

$$x^{\triangle\triangle} \leq x \iff x^{\triangle\triangle} \vee x = x \text{ and } x^{\triangledown\triangledown} \geq x \iff x^{\triangledown\triangledown} \wedge x = x.$$

Thus the conditions (11) and (11') can be written as equations as well. For (12) and (12'), we have

$$x \leq y \Longrightarrow x^\triangle \geq y^\triangle \text{ is equivalent to } (x \wedge y)^\triangle \wedge y^\triangle = y^\triangle$$

and

$$x \leq y \Longrightarrow x^\triangledown \geq y^\triangledown \text{ equivalent to } (x \wedge y)^\triangledown \wedge y^\triangledown = y^\triangledown.$$

Therefore the class of weakly dicomplemented lattices forms a variety. We denote it by WDL. Similarly, the class WCL of weakly complemented lattices and the class DCL of dual weakly complemented lattices are varieties. These classes have been introduced to capture the notion of negation on "formal concepts" [6,12], based on the work of Boole [1].

The following properties are easy to verify:

(14) $y \vee y^\triangle = 1$, $0^\triangle = 1$, $y \wedge y^\triangledown = 0$, $1^\triangledown = 0$ and $y^\triangledown \leq y^\triangle$,
(15) $x^{\triangle\triangle\triangle} = x^\triangle$ and $x \mapsto x^{\triangle\triangle}$ is a kernel operator on L, and
(16) $x^{\triangledown\triangledown\triangledown} = x^\triangledown$ and $x \mapsto x^{\triangledown\triangledown}$ is a closure operator on L.

Before we move to more properties, we give some examples.

(a) The natural examples of weakly dicomplemented lattices are Boolean algebras. In fact if $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra then $(B, \wedge, \vee, ', ', 0, 1)$ (the complementation is duplicated, i.e. $x^\triangle := x' =: x^\triangledown$) is a weakly dicomplemented lattice.
(b) Each bounded lattice can be endowed with a trivial weak dicomplementation by defining $(1, 1)$, $(0, 0)$ and $(1, 0)$ as the dicomplement of 0, 1 and of each $x \notin \{0, 1\}$, respectively.
(c) Concept algebras (see Section 4) form a large class of weakly dicomplemented lattices [6,12].

Theorem 1 (*[7]*). *Weakly complemented lattices are exactly non empty lattices satisfying Eqs.* (11)–(13) *in Definition 2.*

Of course, weakly complemented lattices satisfy Eqs. (11)–(13) in Definition 2. So what we should prove, is that, all lattices satisfying the equations (11)–(13) are bounded.

Proof. Let L be a non empty lattice satisfying Eqs. (11)–(13). Let $x \in L$. We set $1 := x \vee x^\triangle$ and $0 := 1^\triangle$. We are going to prove that 1 and 0 are respectively the greatest and lowest element of L. Let y be an arbitrary element of L. We have

$$1 \geq y \wedge 1 = y \wedge (x \vee x^\triangle) \geq (y \wedge x) \vee (y \wedge x^\triangle) = y, \qquad \text{by (13).}$$